$$\frac{M_{cr}}{M_d} = \frac{33.2}{30.9} > 1. \text{ Hence } (I_e)_d = I_g = 10{,}650 \text{ in.}^4$$

b. Under sustained load

$$\left(\frac{M_{cr}}{M_{sus}}\right)^3 = \left(\frac{33.2}{42.6}\right)^3 = 0.473$$

$$(I_e)_{sus} = (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3] I_{cr} \leq I_g$$ *Eq. (9-8)*

$$= (0.473)(10{,}650) + (1 - 0.473)(3770)$$

$$= 7025 \text{ in.}^4$$

c. Under dead + live load

$$\left(\frac{M_{cr}}{M_{d+\ell}}\right)^3 = \left(\frac{33.2}{54.3}\right)^3 = 0.229$$

$$(I_e)_{d+\ell} = (0.229)(10{,}650) + (1 - 0.229)(3770)$$

$$= 5345 \text{ in.}^4$$

6. Initial or short-time deflections, using Eq. (3): *9.5.2.2* *9.5.2.3*

$$(\Delta_i)_d = \frac{K(5/48)M_d\ell^2}{E_c(I_e)_d} = \frac{(1)(5/48)(30.9)(25)^2(12)^3}{(3320)(10{,}650)} = 0.098 \text{ in.}$$

$K = 1$ for simple spans (see Table 8-3)

$$(\Delta_i)_{sus} = \frac{K(5/48)M_{sus}\ell^2}{E_c(I_e)_{sus}} = \frac{(1)(5/48)(42.6)(25)^2(12)^3}{(3320)(7025)} = 0.205 \text{ in.}$$

$$(\Delta_i)_{d+\ell} = \frac{K(5/48)M_{d+\ell}\ell^2}{E_c(I_e)_{d+\ell}} = \frac{(1)(5/48)(54.3)(25)^2(12)^3}{(3320)(5345)} = 0.344 \text{ in.}$$

$$(\Delta_i)_\ell = (\Delta_i)_{d+\ell} - (\Delta_i)_d = 0.344 - 0.098 = 0.246 \text{ in.}$$

Allowable Deflections (Table 9.5(b)):

Flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$$(\Delta_i)_\ell \leq \frac{\ell}{180} = \frac{300}{180} = 1.67 \text{ in.} > 0.246 \text{ in.} \quad \text{O.K.}$$

Floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$$(\Delta_i)_\ell \le \frac{\ell}{360} = \frac{300}{360} = 0.83 \text{ in.} > 0.246 \text{ in.} \quad \text{O.K.}$$

7. Additional long-term deflections at ages 3 mos. and 5 yrs. (ultimate value):

Combined creep and shrinkage deflections, using Eqs. (9-11) and (4):

Duration	ξ	$\lambda = \frac{\xi}{1 + 50\rho'}$	$(\Delta_i)_{sus}$ in.	$(\Delta_i)_\ell$ in.	$\Delta_{cp} + \Delta_{sh} = \lambda(\Delta_i)_{sus}$ in.	$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell$ in.
5-years	2.0	1.77	0.205	0.246	0.363	0.61
3-months	1.0	0.89	0.205	0.246	0.182	0.43

Separate creep and shrinkage deflections, using Eqs. (5) and (6):

For $\rho = 0.0077$; $\rho' = 0.0026$

For $\rho = 100\rho = 0.77$ and $\rho' = 100\rho' = 0.26$, read $A_{sh} = 0.455$ (Fig. 10-3) and $K_{sh} = 0.125$ for simple spans (Table10-5).

Duration	C_t	$\lambda_{cp} = \frac{0.85C_t}{1+50\rho'}$	$\Delta_{cp} = \lambda_{cp}(\Delta_i)_{sus}$ in.	ε_{sh} in./in.	$\phi_{sh} = \frac{A_{sh}\varepsilon_{sh}}{h}$ 1/in.	$\Delta_{sh} = K_{sh}\phi_{sh}\ell^2$ in.	$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell$ in.
5-years	1.6 (ultimate)	1.20	0.246	400×10^{-6}	$\frac{0.455 \times 400 \times 10^{-6}}{22} = 8.27 \times 10^{-6}$	$\frac{1}{8} \times 8.27 \times 10^{-6} \times (25 \times 12)^2 = 0.093$	0.246+0.093+0.246 = 0.59
3-months	$0.56 \times 1.6 = 0.9$	0.68	0.14	$0.6 \times 400 \times 10^{-6} = 240 \times 10^{-6}$	4.96×10^{-6}	= 0.0558	0.14+0.056+0.246 = 0.44

Allowable Deflection Table 9.5(b):

Roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections (very stringent limitation).

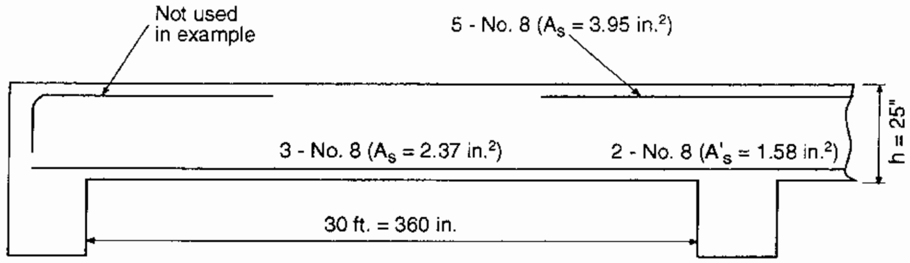
$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \le \frac{\ell}{480} = \frac{300}{480} = 0.63 \text{ in.} \quad \text{O.K. by both methods}$$

Roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections.

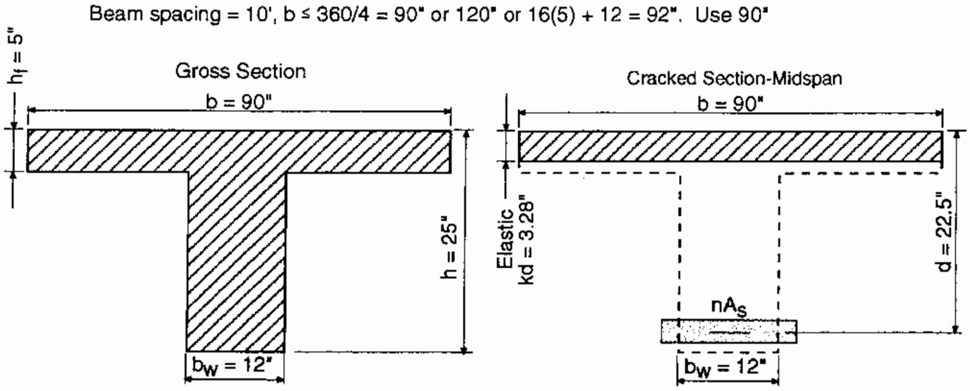
$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \le \frac{\ell}{240} = \frac{300}{240} = 1.25 \text{ in.} \quad \text{O.K. by both methods}$$

Example 10.2—Continuous Nonprestressed T-Beam

Required: Analysis of short-term and ultimate long-term deflections of end-span of multi-span beam shown below.



Beam spacing = 10', b ≤ 360/4 = 90" or 120" or 16(5) + 12 = 92". Use 90"



$nA_s = (11.3)(2.37) = 26.8$ in.²
$\rho = 2.37/(90)(22.5) = 0.00117$
$\rho_w = 2.37/(12)(22.5) = 0.00878$
$\rho' = 0$

Data:

f'_c = 4000 psi (sand-lightweight concrete)
f_y = 50,000 psi
w_c = 120 pcf

Beam spacing = 10 ft
Superimposed Dead Load (not including beam weight) = 20 psf
Live Load = 100 psf (30% sustained)

(A'_s is not required for strength)

Beam will be assumed to be continuous at one end only for h_{min} in Table 9.5(a), for Avg. I_e in Eq. (1), and for K_{sh} in Eq. (6), since the exterior end is supported by a spandrel beam. The end span might be assumed to be continuous at both ends when supported by an exterior column.

Calculations and Discussion	Code Reference

1. Minimum thickness, for members not supporting or attached to partitions or other construction likely to be damaged by large deflections:

$h_{min} = \dfrac{\ell}{18.5}$ *Table 9.5(a)*

Modifying factors = 1.09 for w_c = 120 pcf [footnote (a) Table 9.5(a)]

= 0.9 for f_y = 50,000 psi [footnote (b) Table 9.5(a)]

$$h_{min} = \left(\frac{360}{18.5}\right)(0.90)\,(1.09) = 19.1 \text{ in.} < h = 25 \text{ in.} \quad \text{O.K.}$$

2. Loads and moments:

$w_d = (20 \times 10) + (120)\,(12 \times 20 + 120 \times 5)/144 = 900$ lb/ft

$w_\ell = (100 \times 10) = 1000$ lb/ft

In lieu of a moment analysis, the ACI approximate moment coefficients may be used as follows: Pos. $M = w\ell_n^2/14$ for positive I_e and maximum deflection, Neg. $M = w\ell_n^2/10$ for negative I_e. *8.3.3*

a. Positive Moments

$$\text{Pos. } M_d = \frac{w_d \ell_n^2}{14} = \frac{(0.900)\,(30)^2}{14} = 57.9 \text{ ft-kips}$$

$$\text{Pos. } M_\ell = \frac{(1.000)\,(30)^2}{14} = 64.3 \text{ ft-kips}$$

$\text{Pos. } M_{d+\ell} = 57.9 + 64.3 = 122.2$ ft-kips

$\text{Pos. } M_{sus} = M_d + 0.30 M_\ell = 57.9 + (0.30)\,(64.3) = 77.2$ ft-kips

b. Negative Moments

$$\text{Neg. } M_d = \frac{w_d \ell_n^2}{10} = \frac{(0.900)\,(30)^2}{10} = 81.0 \text{ ft-kips}$$

$$\text{Neg. } M_\ell = \frac{(1.000)\,(30)^2}{10} = 90.0 \text{ ft-kips}$$

$\text{Neg. } M_{d+\ell} = 81.0 + 90.0 = 171.0$ ft-kips

$\text{Neg. } M_{sus} = M_d + 0.30 M_\ell = 81.0 + (0.30)\,(90.0) = 108.0$ ft-kips

3. Modulus of rupture, modulus of elasticity, modular ratio:

$f_r = (0.85)\,(7.5)\sqrt{f'_c} = 6.38\sqrt{4000} = 404$ psi (0.85 for sand lightweight concrete) *Eq. (9-10)*

9.5.2.3(b)

$E_c = w_c^{1.5}\, 33\sqrt{f'_c} = (120)^{1.5}\, 33\sqrt{4000} = 2.74 \times 10^6$ psi *8.5.1*

$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6}{2.74 \times 10^6} = 10.6$$

4. Gross and cracked section moments of inertia:

a. Positive moment section

$y_t = h - (1/2)\,[(b - b_w)\, h_f^2 + b_w h^2]/[(b - b_w)\, h_f + b_w h]$

$= 25 - (1/2)\,[(78)\,(5)^2 + (12)(25)^2]/[(78)\,(5) + (12)\,(25)]$

$= 18.15$ in.

$I_g = (b - b_w)\, h_f^3/12 + b_w h^3/12 + (b - b_w)\, h_f\, (h - h_f/2 - y_t)^2 + b_w h\, (y_t - h/2)^2$

$= (78)\,(5)^3/12 + (12)\,(25)^3/12 + (78)\,(5)\,(25 - 2.5 - 18.15)^2$

$+ (12)\,(25)\,(18.15 - 12.5)^2 = 33{,}390$ in.^{4}

$$B = \frac{b}{nA_s} = \frac{90}{(10.6)\,(2.37)} = 3.58\text{/in.} \quad \text{(Table 10-2)}$$

$$kd = \frac{\sqrt{2dB + 1} - 1}{B} = \frac{\sqrt{(2)\,(22.5)\,(3.58) + 1} - 1}{3.58}$$

$= 3.28$ in. $< h_f = 5$ in.

Hence, treat as a rectangular compression area.

$I_{cr} = bk^3d^3/3 + nA_s\,(d - kd)^2 = (90)\,(3.28)^3/3 + (10.6)\,(2.37)\,(22.5 - 3.28)^2$

$= 10{,}340$ in.^{4}

b. Negative moment section

$$I_g = \frac{12 \times 25^3}{12} = 15{,}625 \text{ in.}^4$$

$I_{cr} = 11{,}185$ in.^{4} (similar to Example 10.1, for b = 12 in., d = 22.5 in., d′ = 2.5 in., $A_s = 3.95$ in.^{2}, $A'_s = 1.58$ in.^{2})

5. Effective moments of inertia, using Eqs. (9-8) and (1):

a. Positive moment section:

$M_{cr} = f_r I_g/y_t = [(404)\,(33{,}390)/(18.15)]/12{,}000 = 61.9$ ft-kips *Eq. (9-9)*

Example 10.2 (cont'd)	Calculations and Discussion	Code Reference

$M_{cr}/M_d = 61.9/57.9 > 1$. Hence $(I_e)_d = I_g = 33{,}390 \text{ in.}^4$

$(M_{cr}/M_{sus})^3 = (61.9/77.2)^3 = 0.515$

$(I_e)_{sus} = (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3] I_{cr} \leq I_g$ — *Eq. (9-8)*

$= (0.515)(33{,}390) + (1 - 0.515)(10{,}340) = 22{,}222 \text{ in.}^4$

$(M_{cr}/M_{d+\ell})^3 = (61.9/122.2)^3 = 0.130$

$(I_e)_{d+\ell} = (0.130)(33{,}390) + (1 - 0.130)(10{,}340) = 13{,}336 \text{ in.}^4$

b. Negative moment section:

$M_{cr} = [(404)(15{,}625)/(12.5)]/12{,}000 = 42.1$ ft-kips — *Eq. (9-9)*

$(M_{cr}/M_d)^3 = (42.1/81.0)^3 = 0.14$

$(I_e)_d = (0.14)(15{,}625) + (1 - 0.14)(11{,}185) = 11{,}808 \text{ in.}^4$ — *Eq. (9-8)*

$(M_{cr}/M_{sus})^3 = (42.1/108.0)^3 = 0.06$

$(I_e)_{sus} = (0.06)(15{,}625) + (1 - 0.06)(11{,}185) = 11{,}448 \text{ in.}^4$ — *Eq. (9-8)*

$(M_{cr}/M_{d+\ell})^3 = (42.1/171.0)^3 = 0.015$

$(I_e)_{d+\ell} = (0.015)(15{,}625) + (1 - 0.015)(11{,}185) = 11{,}251 \text{ in.}^4$ — *Eq. (9-8)*

c. Average inertia values:

Avg. $(I_e) = 0.85 I_m + 0.15 (I_{cont.\ end})$ — *Eq. (1)*

Avg. $(I_e)_d = (0.85)(33{,}390) + (0.15)(11{,}808) = 30{,}153 \text{ in.}^4$

Avg. $(I_e)_{sus} = (0.85)(22{,}222) + (0.15)(11{,}448) = 20{,}606 \text{ in.}^4$

Avg. $(I_e)_{d+\ell} = (0.85)(13{,}336) + (0.15)(11{,}251) = 13{,}023 \text{ in.}^4$

6. Initial or short-time deflections, with midspan I_e and with avg. I_e: — *9.5.2.4*

$$(\Delta_i) = K\left(\frac{5}{48}\right)\frac{M_a \ell^2}{E_c I_e}$$ — *Eq. (3)*

$K = 1.20 - 0.20 M_o/M_a = 1.20 - (0.20)\left(w\ell_n^2/8\right)/\left(w\ell_n^2/14\right) = 0.850$ — *(Table 10-3)*

$$(\Delta_i)_d = \frac{K(5/48) M_d \ell^2}{E_c (I_e)_d} = \frac{(0.85)(5/48)(57.9)(30)^2(12)^3}{(2740)(33{,}390)} = 0.087 \text{ in.}$$

$= 0.096$ in., using avg. $I_e = 30{,}149$ in.^{4}

$$(\Delta_i)_{sus} = \frac{K(5/48)M_{sus}\ell^2}{E_c(I_e)_{sus}} = \frac{(0.85)(5/48)(77.2)(30)^2(12)^3}{(2740)(22{,}222)} = 0.175 \text{ in.}$$

$= 0.188$ in., using avg. $I_e = 20{,}594$ in.^{4}

$$(\Delta_i)_{d+\ell} = \frac{K(5/48)M_{d+\ell}\ell^2}{E_c(I_e)_{d+\ell}} = \frac{(0.85)(5/48)(122.2)(30)^2(12)^3}{(2740)(13{,}336)} = 0.460 \text{ in.}$$

$= 0.472$ in., using avg. $I_e = 13{,}023$ in.^{4}

$$(\Delta_i)_\ell = (\Delta_i)_{d+\ell} - (\Delta_i)_d = 0.460 - 0.087 = 0.373 \text{ in.}$$

$= 0.472 - 0.096 = 0.376$ in., using avg. I_e from Eq. (1)

Allowable deflections Table 9.5(b):

For flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections — $(\Delta_i)_\ell \le \ell/180 = 2.00$ in. > 0.376 in. O.K

For floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections — $(\Delta_i)_\ell \le \ell/360 = 360/360 = 1.00$ in. O.K.

7. Ultimate long-term deflections:

Using ACI Method with combined creep and shrinkage effects:

$$\lambda = \frac{\xi}{1 + 50\rho'} = \frac{2.0 \text{ (ultimate value)}}{1 + 0} = 2.0$$ *Eq. (9-11)*

$$\Delta_{(cp+sh)} = \lambda(\Delta_i)_{sus} = (2.0)(0.175) = 0.350 \text{ in.}$$ *Eq. (4)*

$$\Delta_{(cp+sh)} + (\Delta_i)\ell = 0.350 + 0.373 = 0.723 \text{ in.}$$

$= [2(0.188) + 0.376] = 0.752$ using avg. I_e from Eq. (1).

Using Alternate Method with separate creep and shrinkage deflections:

$$\lambda_{cp} = \frac{0.85C_u}{1 + 50\rho'} = \frac{(0.85)(1.60)}{1 + 0} = 1.36$$

$$\Delta_{cp} = \lambda_{cp}(\Delta_i)_{sus} = (1.36)0.175 = 0.238 \text{ in.}$$ *Eq. (5)*

$= 1.36(0.188) = 0.256$ in., using avg. I_e Eq. (1).

$$\rho = 100\left(\frac{\rho + \rho_w}{2}\right) = 100\left(\frac{2.37}{90 \times 22.5} + \frac{2.37}{12 \times 22.5}\right)/2$$

$$= 100\,(0.00117 + 0.00878)/2 = 0.498\%$$

A_{sh} (from Fig. 10-3) $= 0.555$

$$\phi_{sh} = A_{sh}\frac{(\varepsilon_{sh})_u}{h} = \frac{(0.555)\,(400 \times 10^{-6})}{25} = 8.88 \times 10^{-6}/\text{in.}$$

$$\Delta_{sh} = K_{sh}\phi_{sh}\ell^2 = (0.090)\left(8.88 \times 10^{-6}\right)(30)^2\,(12)^2 = 0.104 \text{ in.}$$ *Eq. (6)*

$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)\ell = 0.238 + 0.104 + 0.373 = 0.715 \text{ in.}$$

$= (0.256 + 0.104 + 0.376) = 0.736.$, using avg. I_e from Eq. (1).

<u>Allowable deflections Table 9.5(b)</u>:

For roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections (very stringent limitation) —

$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \le \frac{\ell}{480} = \frac{360}{480} = 0.75 \text{ in.}$$

All results O.K.

For roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections —

$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \le \frac{\ell}{240} = \frac{360}{240} = 1.50 \text{ in.}$$ All results O.K.

Example 10.3—Slab System Without Beams (Flat Plate)

Required: Analysis of short-term and ultimate long-term deflections of a corner panel.

Data:

Flat plate with no edge beams, designed by Direct Design Method
Slab f'_c = 3000 psi, Column f'_c = 5000 psi, (normal weight concrete)
f_y = 40,000 psi
Square panels—15 × 15 ft center-to-center of columns
Square columns—14 × 14 in., Clear span, ℓ_n = 15 - 1.17 = 13.83 ft
Story height = 10 ft., Slab thickness, h = 6 in.
The reinforcement in the column strip negative moment regions consists of No. 5 bars at 7.5 in. spacing. Therefore, the total area of steel in a 90-in. strip (half the panel length) is given by:

$A_s = (90/7.5)(0.31) = 3.72$ sq. in.

The distance from the compressive side of the slab to the center of the steel is:

d = 4.62 in

Middle Strip reinforcement and d values are not required for deflection computations, since the slab remains uncracked in the middle strips.
Superimposed Dead Load = 10 psf
Live Load = 50 psf
Check for 0% and 40% Sustained Live Load

Calculations and Discussion	Code Reference
1. Minimum thickness:	*9.5.3.2*

From Table 10-6, with Grade 40 steel:

Interior panel $h_{min} = \dfrac{\ell_n}{36} = (13.83 \times 12)/36 = 4.61$ in.

Exterior panel $h_{min} = \dfrac{\ell_n}{33} = (13.83 \times 12)/33 = 5.03$ in.

Since the actual slab thickness is 6 in., deflection calculations are not required; however, as an illustration, deflections will be checked for a corner panel, to make sure that all allowable deflections per Table 9.5(b) are satisfied.

2. Comment on trial design with regard to deflections:

Based on the minimum thickness limitations versus the actual slab thickness, it appears likely that computed deflections will meet most or all of the code deflection limitations. It turns out that all are met.

3. Modulus of rupture, modulus of elasticity, modular ratio:

$$f_r = 7.5\sqrt{f'_c} = 7.5\sqrt{3000} = 411 \text{ psi}$$ *Eq. (9-10)*

$$E_{cs} = w_c^{1.5}\,33\sqrt{f'_c} = (150)^{1.5}\,33\sqrt{3000} = 3.32 \times 10^6 \text{ psi}$$ *8.5.1*

$$E_{cc} = (150)^{1.5}\,33\sqrt{5000} = 4.29 \times 10^6 \text{ psi}$$

$$n = \frac{E_s}{E_{cs}} = \frac{29}{3.32} = 8.73$$

4. Service load moments and cracking moment:

$$w_d = 10 + (150)\,(6.0)/12 = 85.0 \text{ psf}$$

$$(M_o)_d = w_d \ell_2 \ell_n^2/8 = (85.0)\,(15)\,(13.83)^2/8000 = 30.48 \text{ ft-kips}$$

$$(M_o)_{d+\ell} = w_d \ell_2 \ell_n^2/8 = (85.0 + 50.0)\,(15)\,(13.83)^2/8000 = 48.41 \text{ ft-kips}$$

$$(M_o)_{sus} = (85 + 0.4 \times 50)\,(15)(13.83)^2/8000 = 37.65 \text{ ft-kips}$$

The moments are distributed to the ends and centers of the column and middle strips according to the coefficients in the tables of Sections 13.6.3.3, 13.6.4.1, 13.6.4.2 and 13.6.4.4. In this case, the span ratio, ℓ_2/ℓ_1, is equal to 1.0. The multipliers of the panel moment, M_o, that are used to make the distribution in an end span are given in the following table:

	Ext. Negative	Positive	Int. Negative
Total Panel	0.26	0.52	0.70
Col. Strip	(1.0)(0.26)	(0.60)(0.52)	(0.75)(0.70)
Mid. Strip	(1.0-1.0)(0.26)	(1.0-0.60)(0.52)	(1.0-0.75)(0.70)

The resulting moments applied to the external and internal ends and to the center span of the column and middle strips are given in the following tables:

Dead Load Moments, ft-kips

	Ext. Negative	Positive	Int. Negative
Total Panel	7.93	15.85	21.34
Col. Strip	7.93	9.51	16.00
Mid. Strip	0	6.34	5.34

Dead Load + Live Load Moments, ft-kips

	Ext. Negative	Positive	Int. Negative
Total Panel	12.59	25.18	33.89
Col. Strip	12.59	15.10	25.41
Mid. Strip	0	10.07	8.47

Sustained Load Moments, ft-kips (Dead Load + 40% Live Load)

	Ext. Negative	Positive	Int. Negative
Total Panel	9.79	19.58	26.36
Col. Strip	9.79	11.75	19.77
Mid. Strip	0	7.83	6.59

The gross moment of inertia of a panel, referred to as the total equivalent frame moment of inertia is:

$$I_{frame} = \ell_s h^3/12 = (15 \times 12)(6)^3/12 = 3{,}240 \text{ in.}^4$$

For this case, the moment of inertia of a column strip or a middle strip is equal to half of the moment of inertia of the total equivalent frame:

$$I_g = \tfrac{1}{2}(3240) = 1{,}620 \text{ in.}^4$$

The cracking moment for either a column strip or a middle strip is obtained from the standard flexure formula based on the uncracked section as follows:

$$(M_{cr})_{c/2} = (M_{cr})_{m/2} = f_r I_g / y_t = (411)(15 \times 12)(6.0)^3/(4)(12)(3.0)(12{,}000)$$

$$= 9.25 \text{ ft-kips}$$

5. Effective moments of inertia:

A comparison of the tabulated applied moments with the cracking moment shows that the apportioned moment at all locations, except at the interior support of the column strips for the live load and sustained load cases, is less than the cracking moment under the imposed loads. The cracked section moment of inertia is, therefore, only required for the <u>column strips</u> in the negative moment zones. Formulas for computation of the cracked section moment of inertia are obtained from Table 10-2:

$$B = \frac{b}{nA_s} = \frac{\frac{1}{2}(15 \times 12)}{8.73 \times 3.72} = 2.77 \left(\frac{1}{\text{in.}}\right)$$

$$kd = \frac{\sqrt{2\,dB + 1} - 1}{B} = \frac{\sqrt{2 \times 4.62 \times 2.77 + 1} - 1}{2.77} = 1.50 \text{ in.}$$

$$I_{cr} = \frac{b(kd)^3}{3} + nA_s(d - kd)^2 = \frac{90 \times 1.50^3}{3} + 8.73 \times 3.72\,(4.62 - 1.50)^2 = 417 \text{ in.}^4$$

To obtain an equivalent moment of inertia for the cracked location, apply the Branson modification to the moments of inertia for cracked and uncracked sections. The approximate moment of inertia in the cracked sections is given by the general formula in Equation (9-8) of ACI 318. From the tables developed in Section 4 above, the ratios of the dead load plus live load and sustained load moments to the cracking moment are found as follows:

For live dead load plus live load:

$$\frac{M_{cr}}{M_a} = \frac{18.50}{25.41} = 0.728$$

$$\left(\frac{M_{cr}}{M_a}\right)^3 = 0.386$$

and for the sustained load case (dead load plus 40% live load):

$$\frac{M_{cr}}{M_a} = \frac{18.50}{19.77} = 0.936$$

$$\left(\frac{M_{cr}}{M_a}\right)^3 = 0.819$$

The equivalent moment of inertia for the two cases are now computed by Eq. (9-8) of ACI 318:

For dead load plus live load:

$$I_e = (0.386)1620 + (1\text{-}0.386)(417) = 881 \text{ in.}^4$$

For sustained load (dead load + 40% live load):

$$I_e = (0.819)1620 + (1\text{-}0.819)(417) = 1402 \text{ in.}^4$$

Finally, the equivalent moment of inertia for the uncracked sections is just the moment of inertia of the gross section, I_g.

To obtain an average moment of inertia for calculation of deflection, the "end" and "midspan" values are then combined according to Equation (1):

For dead load plus live load:

$$\text{Avg. } I_e = 0.85(1620) + 0.15(881)] = 1509 \text{ in.}^4$$

For sustained load (dead load + 40% live load):

$$\text{Avg. } I_e = 0.85(1620) + 0.15(1402)] = 1587 \text{ in.}^4$$

To obtain the equivalent moment of inertia for the "equivalent frame", which consists of a column and a middle strip, add the average moments of inertia for the respective strips. For the middle strips, the moment of inertia is that of the gross section, I_g, and for the column strips, the average values computed above are used:

For dead load only:

$$(I_e)_{frame} = 1620 + 1620 = 3240 \text{ in.}^4$$

For dead load plus live load:

$$(I_e)_{frame} = 1620 + 1509 = 3129 \text{ in.}^4$$

For dead load plus 40% live load:

$$(I_e)_{frame} = 1620 + 1587 = 3207 \text{ in.}^4$$

Note: In this case, where a corner panel is considered, there is only half of a column strip along the two outer edges. However, the section properties for half a strip are equal to half of those for a full strip; also, the applied moments to the edge strip are half those applied to an interior strip. Consequently, deflections calculated for either a half or for a full column strip are the same. Strictly, these relationships only apply because all panels are of equal dimensions in both directions. If the panels are not square or if adjacent panels are of differing dimensions, additional calculations would be necessary.

6. Flexural stiffness (K_{ec}) of an exterior equivalent column:

$K_b = 0$ (no beams) *R13.7.4*

The stiffness of the equivalent exterior column is determined by combining the stiffness of the upper and lower columns at the outer boundary of the floor with the torsional stiffness offered by a strip of the floor slab, parallel to the edge normal to the direction of the equivalent frame and extending the full panel length between columns. In the case of a corner column, the length is, of course, only half the panel length. The width of the strip is equal to the column dimension normal to the direction of the equivalent frame (ACI 318, R13.7.5).

The column stiffness is computed on the basis of the rotation resulting from application of a moment to the simply supported end of a propped cantilever, M = 4EI/L. In this case the result is:

$$K_c = 4E_{cc}I_c/\ell_c = 4E_{cc}\,[(14)^4/(12)]/[(10)\,(12)] = 106.7E_{cc}$$

Since the columns above and below the slab are equal in dimension, the total stiffness of the columns is twice that of a single column:

$$\Sigma K_c = 2K_c = (2)\,(106.7E_{cc}) = 213.4E_{cc}$$

The torsional stiffness of the slab strip is calculated according to the methodology set out in R13.7.5 of ACI 318, $K_t = \Sigma 9E_{cs}/L_2(1\text{-}c_2/L_2)3$. The cross-sectional torsional constant, C, is defined in Section 13.0 of ACI 318.

$$C = (1 - 0.63\ x/y)\ (x^3y/3) = (1 - 0.636 \times 6.0/14)\ (6.0^3 \times 14/3) = 735.8\ \text{in.}^4$$

$$K_t = \frac{\Sigma 9E_{cs}C}{\ell_2\,(1 - c_2/\ell_2)^3} = \frac{(2)\,(9)\,E_{cs}\,(735.8)}{(15)\,(12)\left(1 - \dfrac{14}{15 \times 12}\right)^3} = 93.9E_{cs}$$

For Ext. Frame, $K_t = 93.9E_{cs}/2 = 47E_{cs}$, $E_{cc} = (4.29/3.32)\ E_{cs} = 1.292E_{cs}$

The equivalent column stiffness is obtained by treating the column stiffness and the torsional member stiffness as springs in series:

$$K_{ec} = \frac{1}{\dfrac{1}{\Sigma K_c} + \dfrac{1}{K_t}} = \frac{E_{cs}}{\left(\dfrac{1}{213.4 \times 1.292}\right) + \left(\dfrac{1}{93.9}\right)} = 70E_{cs} = 19{,}370\ \text{ft-kips/rad}$$

$$\text{For Ext. Frame, } K_{ec} = \frac{E_{cs}}{\left(\dfrac{1}{213.4 \times 1.292}\right) + \left(\dfrac{1}{47.0}\right)} = 40.1E_{cs} = 11{,}090\ \text{ft-kips/rad}$$

7. Deflections, using Eqs. (7) to (14):

$$\text{Fixed } \Delta_{frame} = w\ell_2\ell^4/384E_{cs}I_{frame}$$ *Eq. (10)*

$$(\text{Fixed } \Delta_{frame})_{d,d+\ell} = \frac{(85.0 \text{ or } 135.0 \text{ or } 105.0)\,(15)^5(12)^3}{(384)\left(3.32 \times 10^6\right)(3240 \text{ or } 3129 \text{ or } 3207)}$$

$$= 0.027\ \text{in., } 0.044\ \text{in.; } 0.034$$

$$\text{Fixed } \Delta_{c,m} = (LDF)_{c,m}\ (\text{Fixed } \Delta_{frame})\ (I_{frame}/I_{c,m})$$ *Eq. (11)*

These deflections are distributed to the column and middle strips in the ratio of the total applied moment to the beam stiffness (M/EI) of the respective strips to that of the complete frame. As shown in Step 4 above, the fraction of bending moment apportioned to the column or middle strips varies between the ends and the midspan. Therefore, in approximating the deflections by this method, the average moment allocation fraction (Lateral Distribution Factor - LDF) is used. In addition, since the equivalent moment of inertia changes whenever the cracking moment is exceeded, an average moment of inertia is utilized. This average moment of inertia is computed on the basis of Equation (9-8) from ACI 318 and Eq. (1) of this chapter. Finally, since the modulus of elasticity is constant throughout the slab, the term E occurs in both the numerator and the denominator and is, therefore, omitted. The LDFs are calculated as follows:

For the column strip:

$$LDF_c = \frac{1}{2}\left[\frac{1}{2}(M^-_{int} + M^-_{ext}) + M^+\right] = \frac{1}{2}\left[\frac{1}{2}(0.75 + 1.00) + 0.60\right] = 0.738$$

For the middle strip:

$$LDF_m = 1 - LDF_c = 0.262$$

$(\text{Fixed } \Delta_c)_d = (0.738)(0.027)(2) = 0.040$ in.

$(\text{Fixed } \Delta_c)_{d+\ell} = (0.738)(0.044)(3129/1509) = 0.067$ in.

$(\text{Fixed } \Delta_c)_\ell = 0.067 - 0.040 = 0.027$ in.

$(\text{Fixed } \Delta_c)_{sus} = (0.738)(0.034)(3207/1587) = 0.051$ in

$(\text{Fixed } \Delta_m)_d = (0.262)(0.027)(2) = 0.014$ in.

$(\text{Fixed } \Delta_m)_{d+\ell} = (0.262)(0.044)(3129/1620) = 0.022$ in.

$(\text{Fixed } \Delta_m)_\ell = 0.022 - 0.014 = 0.008$ in.

$(\text{Fixed } \Delta_m)_{sus} = (0.0.262)(0.034)(3207/1587) = 0.018$ in

In addition to the fixed end displacement found above, an increment of deflection must be added to each due to the actual rotation that occurs at the supports. The magnitude of the increment is equal to qL/8. The rotations, q, are determined as the net moments at the column locations divided by the effective column stiffnesses. In this case, the column strip moment at the corner column of the floor is equal to half of 100% of 0.26 x M_o (ACI 318, Sec. 13.6.3.313 and Sec.13.6.4.2). Because the column strip at the edge of the floor is only half as wide as an interior column strip, only half of the apportioned moment acts. The net moments at other columns are either quite small or zero. Therefore they are neglected. The net moments on a corner column for the three loading cases are:

$(M_{net})_d = \frac{1}{2} \times 0.26 \times 1.00 \times (M_o)_d = \frac{1}{2}\,[(0.26)(1.00)](30.48) = 3.96$ ft-kips

$(M_{net})_{d+\ell} = \frac{1}{2} \times 0.26 \times 1.00 \times (M_o)_{d+\ell} = \frac{1}{2}\,[(0.26)(1.00)](48.41) = 6.29$ ft-kips

$(M_{net})_{sus} = \frac{1}{2} \times 0.26 \times 1.00 \times (M_o)_{sus} = \frac{1}{2}\,[(0.26)(1.00)](37.65) = 4.89$ ft-kips

For both column and middle strips,

End $\theta_d = (M_{net})_d/\text{avg. } K_{ec} = 3.96/11{,}090 = 0.000357$ rad *Eq. (12)*

End $\theta_{d+\ell} = 6.29/11{,}090 = 0.000567$ rad

End θ_{sus} = 4.89/11090 = 0.000441 rad

$\Delta\theta$ = (End θ) (ℓ/8) $(I_g/I_e)_{frame}$ *Eq. (14)*

$(\Delta\theta)_d$ = (0.000357) (15) (12) (1)/8 = 0.008 in.

$(\Delta\theta)_{d+\ell}$ = (0.000567) (15) (12) (1620/1509)/8 = 0.014 in.

$(\Delta\theta)_\ell$ = 0.014 - 0.008 = 0.006 in.

$(\Delta\theta)_{sus}$ = (0.000441)(15)(12)/8 = 0.010 in.

The deflections due to rotation calculated above are for column strips. The deflections due to end rotations for the middle strips will be assumed to be equal to that in the column strips. Therefore, the strip deflections are calculated by the general relationship:

$\Delta_{c,m}$ = Fixed $\Delta_{c,m}$ + $(\Delta\theta)$ *Eq.(9)*

$(\Delta_c)_d$ = 0.040 + 0.008 = 0.048 in.

$(\Delta_m)_d$ = 0.014 + 0.008 = 0.022 in.

$(\Delta_c)_\ell$ = 0.027 + 0.006 = 0.033 in.

$(\Delta_m)_\ell$ = 0.008 + 0.006 = 0.014 in.

$(\Delta_c)_{sus}$ = 0.051 + (0.010) = 0.061 in.

$(\Delta_m)_{sus}$ = 0.018 + (0.010) = 0.028 in.

$\Delta = \Delta_{cx} + \Delta_{my}$ = midpanel deflection of corner panel *Eq. (7)*

$(\Delta_i)_d$ = 0.048 + 0.022 = 0.070 in.

$(\Delta_i)_\ell$ = 0.033 + 0.014 = 0.047 in.

$(\Delta_i)_{sus}$ = 0.061 + 0.028 = 0.089 in.

The long term deflections may be calculated using Eq. (9-11) of ACI 318 (Note: ρ' = 0):

For dead load only:

$(\Delta_{cp+sh})_d = 2.0 \times (\Delta_i)_d$ = (2)(0.070) = 0.140 in.

For sustained load (dead load + 40% live load)

$(\Delta_{cp+sh})_{sus} = 2.0 \times (\Delta_i)_{sus} = (2)(0.109) = 0.218$ in.

The long term deflection due to sustained load plus live load is calculated as:

$(\Delta_{cp+sh})_{sus} + (\Delta_i)_\ell = 0.218 + 0.047 = 0.265$ in.

These computed deflections are compared with the code allowable deflections in Table 9.5(b) as follows:

Flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$(\Delta_i)_\ell \leq (\ell_n \text{ or } \ell)/180 = (13.83 \text{ or } 15)(12)/180 = 0.92$ in. or 1.00 in., versus 0.047 in. O.K.

Floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$(\Delta_i)_\ell \leq (\ell_n \text{ or } \ell)/360 = 0.46$ in. or 0.50 in., versus 0.047 in. O.K.

Roof or floor construction supporting or attached to non-structural elements likely to be damaged by large deflections—

$\Delta_{(cp+sh)} + (\Delta_i)_\ell \leq (\ell_n \text{ or } \ell)/480 = 0.35$ in. or 0.38 in., versus 0.265 in. O.K.

Roof or floor construction supporting or attached to non-structural elements not likely to be damaged by large deflections—

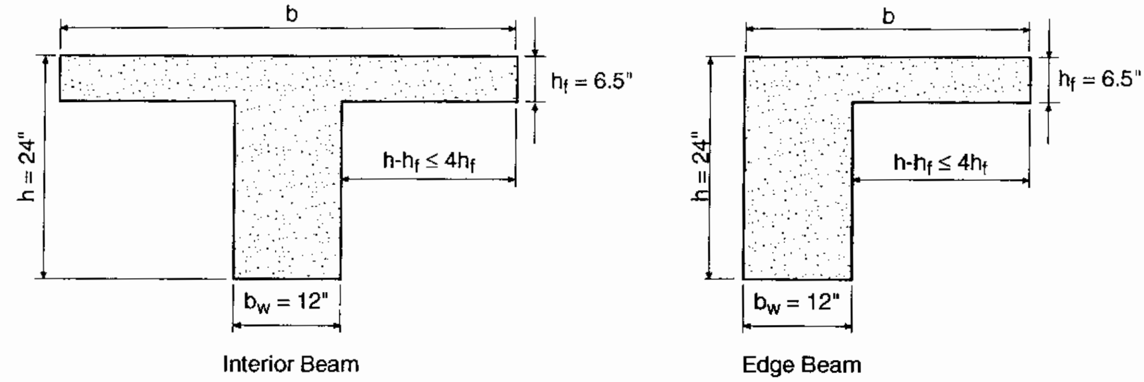
$\Delta_{(cp+sh)} + (\Delta_i)_\ell \leq (\ell_n \text{ or } \ell)/240 = 0.69$ in. or 0.75 in., versus 0.265 in. O.K.

All computed deflections are found to be satisfactory in all four categories.

Example 10.4—Two-Way Beam Supported Slab System

Required: Minimum thickness for deflection control

Data:



f_y = 60,000 psi, slab thickness h_f = 6.5 in.
Square panels—22 × 22 ft center-to-center of columns
All beams—b_w = 12 in. and h = 24 in. ℓ_n = 22 - 1 = 21 ft
It is noted that f'_c and the loading are not required in this analysis.

Calculations and Discussion	Code Reference

1. Effective width b and section properties, using Table 10-2:

   a. Interior Beam

   $I_s = (22)(12)(6.5)^3/12 = 6040 \text{ in.}^4$

   $h - h_f = 24 - 6.5 = 17.5 \text{ in.} \le 4h_f = (4)(6.5) = 26 \text{ in.}$ O.K.

   Hence, $b = 12 + (2)(17.5) = 47$ in.

   $y_t = h - (1/2)[(b - b_w)h_f^2 + b_wh^2]/[(b - b_w)h_f + b_wh]$

   $= 24 - (1/2)[(35)(6.5)^2 + (12)(24)^2]/[(35)(6.5) + (12)(24)]$

   $= 15.86$ in.

   $I_b = (b - b_w)h_f^3/12 + b_wh^3/12 + (b - b_w)h_f(h - h_f/2 - y_t)^2 + b_wh(y_t - h/2)^2$

   $= (35)(6.5)^3/12 + (12)(24)^3/12 + (35)(6.5)(24 - 3.25 - 15.86)^2 +$

   $(12)(24)(15.86 - 12)^2 = 24{,}360 \text{ in.}^4$

   $\alpha_f = E_{cb}I_b/E_{cs}I_s = I_b/I_s = 24{,}360/6040 = 4.03$

b. Edge Beam

$I_s = (11)(12)(6.5)^3/12 = 3020 \text{ in.}^4$

$b = 12 + (24 - 6.5) = 29.5 \text{ in.}$

$y_t = 24 - (1/2)[(17.5)(6.5)^2 + (12)(24)^2]/[(17.5)(6.5) + (12)(24)] = 14.48 \text{ in.}$

$I_b = (17.5)(6.5)^3/12 + (12)(24)^3/12 + (17.5)(6.5)(24 - 3.25 - 14.48)^2 +$

$(12)(24)(14.48 - 12)^2 = 20,470 \text{ in.}^4$

$\alpha_f = I_b/I_s = 20,470/3020 = 6.78$

α_{fm} and β values:

α_{fm} (average value of α_f for all beams on the edges of a panel):

Interior panel — $\alpha_{fm} = 4.03$

Side panel — $\alpha_{fm} = [(3)(4.03) + 6.78]/4 = 4.72$

Corner panel — $\alpha_{fm} = [(2)(4.03) + (2)(6.78)]/4 = 5.41$

For square panels, β = ratio of clear spans in the two directions = 1

2. Minimum thickness: *9.5.3.3*

Since $\alpha_{fm} > 2.0$ for all panels, Eq. (9-13) applies.

$$h_{min} = \frac{\ell_n\left(0.8 + \frac{f_y}{200,000}\right)}{36 + 9\beta}$$ *Eq. (9-13)*

$$= \frac{(21 \times 12)\left(0.8 + \frac{60,000}{200,000}\right)}{36 + 9(1)} = 6.16 \text{ in.} \quad \text{(all panels)}$$

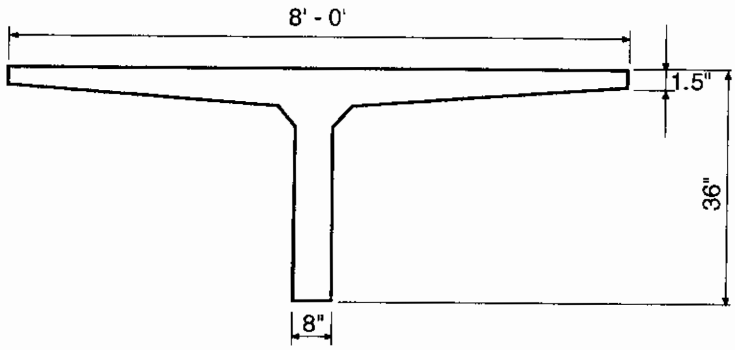
Hence, the slab thickness of 6.5 in. > 6.16 in. is satisfactory for all panels, and deflections need not be checked.

Example 10.5—Simple-Span Prestressed Single T-Beam

Required: Analysis of short-term and ultimate long-term camber and deflection.

Data:

8ST36 (Design details from PCI Handbook 3rd Edition, 1985)
Span = 80 ft, beam is partially cracked
$f'_{ci} = 3500$ psi, $f'_c = 5000$ psi (normal weight concrete)
$f_{pu} = 270{,}000$ psi
$E_p = 27{,}000{,}000$ psi
14 - 1/2 in. dia. depressed (1 Pt.) strands
4 - 1/2 in. dia. nonprestressed strands
(Assume same centroid when computing I_{cr})
$P_i = (0.7)(14)(0.153)(270) = 404.8$ kips
$P_o = (0.90)(404.8) = 364$ kips
$P_e = (0.78)(404.8) = 316$ kips
$e_e = 11.15$ in., $e_c = 22.51$ in.
$y_t = 26.01$ in., $A_g = 570$ in.^{2}, $I_g = 68{,}920$ in.^{4}
Self weight, $w_o = 594$ lb/ft
Superimposed DL, $w_s = (8)(10 \text{ psf}) = 80$ lb/ft is applied at age 2 mos ($\beta_s = 0.76$ in Term (6) of Eq. (15))
Live load, $w_\ell = (8)(51 \text{ psf}) = 408$ lb/ft
Capacity is governed by flexural strength



Calculations and Discussion	Code Reference

1. Span-depth ratios (using PCI Handbook):

   Typical span-depth ratios for single T beams are 25 to 35 for floors and 35 to 40 for roofs, versus (80)(12)/36 = 27, which indicates a relatively deep beam. It turns out that all allowable deflections in Table 9.5(b) are satisfied.

2. Moments for computing deflections:

$$M_o = \frac{w_o \ell^2}{8} = \frac{(0.594)(80)^2}{8} = 475 \text{ ft-kips}$$

($\times$ 0.96 = 456 ft-kips at 0.4ℓ for computing stresses and I_e— tendons depressed at one point)

$$M_s = \frac{w_s\ell^2}{8} = \frac{(0.080)\,(80)^2}{8} = 64 \text{ ft-kips } (61 \text{ ft-kips at } 0.4\ell)$$

$$M_\ell = \frac{w_\ell\ell^2}{8} = \frac{(0.408)\,(80)^2}{8} = 326 \text{ ft-kips } (313 \text{ ft-kips at } 0.4\ell)$$

3. Modulus of rupture, modulus of elasticity, moment of inertia:

$$f_r = 7.5\sqrt{f'_c} = 7.5\sqrt{5000} = 530 \text{ psi}$$ *Eq. (9-10)*

$$E_{ci} = w_c^{1.5}\, 33\sqrt{f'_{ci}} = (150)^{1.5}\, 33\sqrt{3500} = 3.59 \times 10^6 \text{ psi}$$ *8.5.1*

$$E_c = w_c^{1.5}\, 33\sqrt{f'_c} = (150)^{1.5}\, 33\sqrt{5000} = 4.29 \times 10^6 \text{ psi}$$

$$n = \frac{E_p}{E_c} = \frac{27 \times 10^6}{4.29 \times 10^6} = 6.3$$

The moment of inertia of the cracked section, at 0.4ℓ, can be obtained by the approximate formula given in Eq. 4.8.2 of the PCI Handbook:

$$I_{cr} = nA_{st}d^2\left(1 - 1.6\sqrt{n\rho}\right) = (6.3)\,(18 \times 0.153)\,(30.23)^2\left(1 - 1.6\sqrt{6.3 \times 0.000949}\right)$$

$$= 13{,}890 \text{ in}^4 \text{ (at } 0.4\ell)$$

It may be shown that the cracked section moment of inertia calculated by the formulas given in Table 10.2 is very close to the value obtained by the approximate method shown above. The results differ by approximately 1%; therefore either method is suitable for this case.

4. Determination of classification of beams

In order to classify the beam according to the requirements of ACI Section 18.3.3, the maximum flexural stress is calculated and compared to the modulus rupture to determine its classification. The classifications are defined as follows:

Class U: $f_t \le 7.5\sqrt{f'_c}$

Class T: $7.5\sqrt{f'_c} < f_t \le 12\sqrt{f'_c}$

Class C: $f_t > 12\sqrt{f'_c}$

The three classes refer to uncracked (U), transition (T) and cracked (C) behavior.

The maximum tensile stress due to service loads and prestressing forces are calculated by the standard formula for beams subject to bending moments and axial loads. It may be shown that the maximum bending stresses in a prestressed beam occur at approximately 0.4ℓ. In the following, the bending moments are those that occur at 0.4ℓ. The eccentricity of the prestressing force at 0.4ℓ, e = 20.24 in, is obtained by linear interpolation between the end eccentricity, e_e = 11.15 in and that at the center, e_e = 22.51. The calculation proceeds as follow:

$$M_{tot} = M_d + M_\ell$$

$$f_t = \frac{M_{tot}y_t}{I_g} - \frac{P_e e \cdot y_t}{I_g} - \frac{P_e}{A_g}$$

$$= [(456 + 61 + 313)(12) - (316)(20.24)]\,[(26.01)/68{,}920] - 316/570$$

$$f_t = 791 \text{ psi}$$

Check the ratio of calculated tensile stress to the square root of f'_c:

$$\frac{f_t}{\sqrt{f'_c}} = \frac{791}{\sqrt{5000}} = 11.2$$

The ratio is between 7.5 and 12, therefore according to the definitions of Section 18.3.3 of ACI 318, the beam classification is T, transition region. Table R18.3.3 requires that deflections for this classification be based on the cracked section properties assuming bi-linear moment deflection behavior; Section 9.5.4.2 of the code allows either a bi-linear moment-deflection approach or calculation of deflections on the basis of an equivalent moment of inertia determined according to Eq. (9-8).

5. Determine live load moment that causes first cracking:

Check the tensile stress due to dead load and prestressing forces only. As noted previously, the maximum tensile stresses occur at approximately 0.4ℓ:

$$f_t = [(456 + 61)(12) - (316)(20.24)]\,[(26.01)/68920] - 316/570 = -627 \text{ psi}$$

Since the sign is negative, compressive stress is indicated. Therefore, the section is uncracked under the dead load plus prestressing forces and dead load deflections can be based on the moment of inertia of the gross concrete cross section. It was shown above that the maximum tensile stress due to combined dead load plus live load equals 791 psi, which exceeds the modulus of rupture, f_r = 530 psi

Therefore, the live load deflections must be computed on the basis of a cracked section analysis because the behavior is inelastic after the addition of full live load. In particular, Table R18.3.3 of ACI 318 requires that bilinear behavior be utilized to determine deflections in such cases, however. Section 9.5.4.2 permits deflections to be computed either on the basis of bilinear behavior or on the basis of an effective moment of inertia.

In order to calculate the deflection assuming bilinear behavior, it is first necessary to determine the fraction of the total live load that causes first cracking. That is, to find the portion of live load that will just produce a maximum tensile stress equal to f_r. The desired value of live load moment can be obtained by re-arranging the equation used above to determine the total tensile stress (for classification), and setting the tensile stress equal to f_r. The moment value is obtained as follows (Note: Quantities calculated at 0.4ℓ):

$$\text{Live Load Cracking Moment} = \frac{f_r I_g}{y_t} + P_e e + \frac{P_e I_g}{A_g y_t} - M_d$$

= (530)68920)/(12000)(26.01) + 316(20.24/12) +[(316/570)(68920/26.01)]/12 - 517

= 117 + 533 +122 - 517

= 255 ft-kips

The fraction of the live load cracking moment to the total live load moment is:

255/313 = 0.815

6. Camber and Deflection, using Eq. (15):

Term (1) — $\Delta_{po} = \dfrac{P_o(e_c - e_e)\ell^2}{12E_{ci}I_g} + \dfrac{P_o e_e \ell^2}{8E_{ci}I_g}$ (from PCI Handbook for single point depressed strands)

$$= \frac{(364)(22.51 - 11.15)(80)^2(12)^2}{(12)(3590)(68{,}920)} + \frac{(364)(11.15)(80)^2(12)^2}{(8)(3590)(68{,}920)}$$

= 3.17 in.

Term (2) — $\Delta_o = \dfrac{5M_o\ell^2}{48E_{ci}I_g} = \dfrac{(5)(475)(80)^2(12)^3}{(48)(3590)(68{,}590)} = 2.21$ in.

Term (3) — $k_r = 1/[1 + (A_s/A_{ps})] = 1/[1 + (4/14)] = 0.78$

$$\left[-\frac{\Delta P_u}{P_o} + (k_r C_u)\left(1 - \frac{\Delta P_u}{2P_o}\right)\right]\Delta_{po}$$

The increment in prestressing force is:

$\Delta P_u = P_o - P_e = 364 - 316 = 48$ kips

It follows that:

$\Delta P_u / P_o = 48/364 = 0.13$

Therefore, the deflection is:

$$= [-0.13 + (0.78 \times 2.0)(1 - 0.065)](3.17) = 4.21 \text{ in.}$$

Term (4) — $(k_r C_u)\,\Delta_o = (0.78)(2.0)(2.21) = 3.45$ in.

$$\text{Term (5) — } \Delta_s = \frac{5 M_s \ell^2}{48 E_c I_g} = \frac{(5)(64)(80)^2(12)^3}{(48)(4290)(68{,}920)} = 0.25 \text{ in.}$$

Term (6) — $(\beta_s k_r C_u)\,\Delta_s = (0.76)(0.78)(1.6)(0.25) = 0.24$ in.

Term (7) – Initial deflection due to live load.

The ratio of live load cracking moment to total live load moment was found previously. To calculate the deflection according to bi-linear behavior, the deflection due to the portion of the live load below the cracking value is based on the gross moment of inertia; the deflection due to the remainder of the live load is based on the cracked section moment of inertia. Also, the deflections are based on moments at the center of the span even though the moment that caused initial cracking was evaluated at 0.4ℓ.

The deflection formula used is the standard expression:

$$\Delta = \frac{5\,ML^2}{48\,EI}$$

For the portion of the live load applied below the cracking moment load, the value of M is the value calculated above, 255 ft-kips and the moment of inertia is that of the gross section:

$$\Delta_{\ell 1} = 5(255)(80)^2(12)^3/48(3590)(68590) = 1.19 \text{ in}$$

Deflection due to the remainder of the live load is calculated similarly, with a moment of 313-255 = 58 ft-kips and the cracked section moment of inertia, 13,890 in^4:

$$\Delta_{\ell 2} = 5(58)(80)^2(12)^3/48(3590)(13{,}890) = 1.34 \text{ in}$$

The total live load deflection is the sum of the previous two components:

$$\Delta_\ell = 1.19 + 1.34 = 2.53 \text{ in.}$$

It can be verified by a separate calculation that the deflection based on the full live load moment and the effective moment of inertia, calculated by Eq. 9-8 of ACI 318, is slightly less than that calculated here on the basis of a bi-linear moment-deflection relationship.

Combined results and comparisons with code limitations

$$\Delta_u = \underset{(1)}{-3.17} + \underset{(2)}{2.21} \underset{(3)}{-4.21} + \underset{(4)}{3.45} + \underset{(5)}{0.25} + \underset{(6)}{0.24} + \underset{(7)}{2.53} = 1.30 \text{ in.} \downarrow$$ *Eq. (15)*

Initial Camber $= \Delta_{po} - \Delta_o = 3.17 - 2.21 = 0.96$ in. $\uparrow$ versus 1.6 in. at erection in PCI Handbook

Residual Camber $= \Delta_\ell - \Delta_u = 2.53 - 0.87 = 1.66$ in. $\uparrow$ versus 1.1 in.

Time-Dependent plus Superimposed Dead Load and Live Load Deflection

$= -4.21 + 3.45 + 0.25 + 0.24 + 2.53 = 2.26$ in. or

$= \Delta_u - \left(\Delta_o - \Delta_{po}\right) = 0.87 - (-0.96) = 2.26$ in. $\downarrow$

These computed deflections are compared with the allowable deflections in Table 9.5(b) as follows:

$\ell/180 = (80)(12)/180 = 5.33$ in. versus $\Delta_\ell = 2.53$ in. O.K.

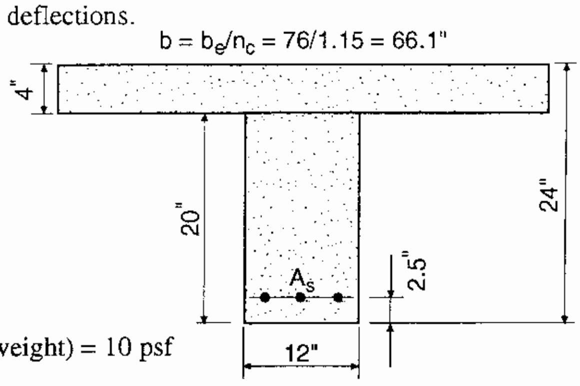
$\ell/360 = (80)(12)/360 = 2.67$ in. versus $\Delta_\ell = 2.53$ in. O.K.

$\ell/480 = (80)(12)/480 = 2.00$ in. versus Time-Dep. etc. $= 2.26$ in. O.K.

Note that the long term deflection occurring after attachment of non-structural elements (2.26 in) exceeds the L/480 limit. It actually meets L/425. Since the L/480 limit only applies in case of *nonstructural elements likely to be damaged by large deflections*, the particular use of the beam would have to be considered in order to make a judgment on the acceptability of the computed deflections. Refer to the footnotes following Table 9.5(b) of ACI 318.

Example 10.6—Unshored Nonprestressed Composite Beam

Required: Analysis of short-term and ultimate long-term deflections.



Data:

Normal weight concrete
Slab $f'_c = 3000$ psi
Precast beam $f'_c = 4000$ psi
$f_y = 40,000$ psi
A_s = 3-No. 9 = 3.00 in.^{2}
E_s = 29,000,000 psi
Superimposed Dead Load (not including beam and slab weight) = 10 psf
Live Load = 75 psf (20% sustained)
Simple span = 26 ft = 312 in., spacing = 8 ft = 96 in.
b_e = 312/4 = 78.0 in., or spacing = 96.0 in., or 16(4) + 12 = 76.0 in.

Calculations and Discussion	Code Reference

1. Minimum thickness for members not supporting or attached to partitions or other construction likely to be damaged by large deflections:

$$h_{min} = \left(\frac{\ell}{16}\right)(0.80 \text{ for } f_y) = \left(\frac{312}{16}\right)(0.80) = 15.6 \text{ in.} < h = 20 \text{ in. or } 24 \text{ in.}$$ *Table 9.5(a)*

2. Loads and moments:

$$w_1 = (10 \text{ psf})(8) + (150 \text{ pcf})(96)(4)/144 = 480 \text{ lb/ft}$$

$$w_2 = (150 \text{ pcf})(12)(20)/144 = 250 \text{ lb/ft}$$

$$w_\ell = (75 \text{ psf})(8) = 600 \text{ lb/ft}$$

$$M_1 = w_1\ell^2/8 = (0.480)(26)^2/8 = 40.6 \text{ ft-kips}$$

$$M_2 = w_2\ell^2/8 = (0.250)(26)^2/8 = 21.1 \text{ ft-kips}$$

$$M_\ell = w_\ell\ell^2/8 = (0.600)(26)^2/8 = 50.7 \text{ ft-kips}$$

3. Modulus of rupture, modulus of elasticity, modular ratio:

$$(E_c)_1 = w_c^{1.5}\, 33\sqrt{f'_c} = (150)^{1.5}\, 33\sqrt{3000} = 3.32 \times 10^6 \text{ psi}$$ *8.5.1*

$$(f_r)_2 = 7.5\sqrt{f'_c} = 7.5\sqrt{4000} = 474 \text{ psi}$$ *Eq. (9-10)*

$$(E_c)_2 = (150)^{1.5}\, 33\sqrt{4000} = 3.83 \times 10^6 \text{ psi}$$ *8.5.1*

$$n_c = \frac{(E_c)_2}{(E_c)_1} = \frac{3.83}{3.32} = 1.15$$

$$n = \frac{E_s}{(E_c)_2} = \frac{29}{3.83} = 7.56$$

4. Gross and cracked section moments of inertia, using Table 10-2:

Precast Section

$I_g = (12)(20)^3/12 = 8000 \text{ in.}^4$

$B = b/(nA_s) = 12/(7.56)(3.00) = 0.529/\text{in.}$

$kd = \left(\sqrt{2dB + 1} - 1\right)/B = \left[\sqrt{(2)(17.5)(0.529) + 1} - 1\right]/0.529 = 6.46 \text{ in.}$

$I_{cr} = bk^3d^3/3 + nA_s(d - kd)^2 = (12)(6.46)^3/3 + (7.56)(3.00)(17.5 - 6.46)^2 = 3840 \text{ in.}^4$

Composite Section

$y_t = h - (1/2)[(b - b_w)h_f^2 + b_wh^2]/[(b - b_w)h_f + b_wh]$

$= 24 - (1/2)[(54.1)(4)^2 + (12)(24)^2]/[(54.1)(4) + (12)(24)] = 16.29 \text{ in.}$

$I_g = (b - b_w)h_f^3/12 + b_wh^3/12 + (b - b_w)h_f(h - h_f/2 - y_t)^2 + b_wh(y_t - h/2)^2$

$= (54.1)(4)^3/12 + (12)(24)^3/12 + (54.1)(4)(24 - 2 - 16.29)^2$

$+ (12)(24)(16.29 - 12)^2 = 26{,}470 \text{ in.}^4$

$B = b/(nA_s) = 66.1/(7.56)(3.00) = 2.914$

$kd = \left(\sqrt{2dB + 1} - 1\right)/B = \left[\sqrt{(2)(21.5)(2.914) + 1} - 1\right]/2.914$

$= 3.51 \text{ in.} < h_f = 4 \text{ in.}$ Hence, treat as a rectangular compression area.

$I_{cr} = bk^3d^3/3 + nA_s(d - kd)^2 = (66.1)(3.51)^3/3 + (7.56)(3.00)(21.5 - 3.51)^2$

$= 8295 \text{ in.}^4$

$I_2/I_c = [(I_2/I_c)_g + (I_2/I_c)_{cr}]/2 = [(8000/26{,}470) + (3840/8295)]/2 = 0.383$

5. Effective moments of inertia, using Eq. (9-8):

For Term (1), Eq. (19)—Precast Section,

$M_{cr} = f_rI_g/y_t = (474)(8000)/(10)(12{,}000) = 31.6$ ft-kips *Eq. (9-9)*

$M_{cr}/M_2 = 31.6/21.1 > 1.$ Hence $(I_e)_2 = I_g = 8000 \text{ in.}^4$

For Term (6), Eq. (19)—Precast Section,

$[M_{cr}/(M_1 + M_2)]^3 = [31.6/(40.6 + 21.1)]^3 = 0.134$

$(I_e)_{1+2} = (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3] I_{cr} \leq I_g$ *Eq. (9-8)*

$= (0.134)(8000) + (1 - 0.134)(3840) = 4400 \text{ in.}^4$

6. Deflection, using Eq. (19):

Term (1) — $(\Delta_i)_2 = \dfrac{K(5/48) M_2 \ell^2}{(E_c)_2 (I_e)_2} = \dfrac{(1)(5/48)(21.1)(26)^2 (12)^3}{(3830)(8000)} = 0.084$ in.

Term (2) — $k_r = 0.85$ (no compression steel in precast beam).

$0.77k_r(\Delta_i)_2 = (0.77)(0.85)(0.084) = 0.055$ in.

Term (3) — $0.83k_r(\Delta_i)_2 \dfrac{I_2}{I_c} = (0.83)(0.85)(0.084)(0.383) = 0.023$ in.

Term (4) — $K_{sh} = 1/8$. Precast Section: $\rho = (100)(3.00)/(12)(17.5) = 1.43\%$

From Fig. 8-3, $A_{sh} = 0.789$

$\phi_{sh} = A_{sh}(\varepsilon_{sh})_u/h = (0.789)\left(400 \times 10^{-6}\right)/20 = 15.78 \times 10^{-6}/\text{in.}$

$\Delta_{sh} = K_{sh}\phi_{sh}\ell^2 = (1/8)\left(15.78 \times 10^{-6}\right)(26)^2 (12)^2 = 0.192$ in.

The ratio of shrinkage strain at 2 months to the ultimate is 0.36 per Table 2.1 of Ref. 10.4 Therefore the shrinkage deflection of the precast beam at 2 months is:

$0.36\Delta_{sh} = (0.36)(0.192) = 0.069$ in.

Term (5) — $0.64\Delta_{sh} \dfrac{I_2}{I_c} = (0.64)(0.192)(0.383) = 0.047$ in.

Term (6) — $(\Delta_i)_1 = \dfrac{K(5/48)(M_1 + M_2)\ell^2}{(E_c)_2 (I_e)_{1+2}} - (\Delta_i)_2$

$= \dfrac{(1)(5/48)(40.6 + 21.1)(26)^2 (12)^3}{(3830)(4400)} - 0.088 = 0.358$ in.

Example 10.6 (cont'd)	Calculations and Discussion	Code Reference

Term (7) —Creep deflection of the composite beam due to slab dead load. The slab is cast at 2 months. Therefore, the fraction of the creep coefficient, C_u, is obtained my multiplying the value under standard conditions of 1.60 by a b_s value of 0.89 (See explanation of Term (6) in Eq. (15)). The total creep of the beam is reduced by the ratio of the moment of inertia of the beam to the moment of inertia of the composite section. k_r is, as before, taken as 0.85:

$$(0.89)(1.60)k_r(\Delta_i)_1 \frac{I_2}{I_c} = (0.89)\,(1.60)\,(0.85)\,(0.358)\,(0.383) = 0.166 \text{ in.}$$

Term (8) —Due to the fact that the beam and the slab were cast at different times, there will be some contribution to the total deflection due to the tendency of the two parts to creep and shrink at different rates. It is noted in Table 2.1 of ACI 435R-95 (Ref. 10.4) that the creep and shrinkage at a time of 2 months is almost half of the total. Consequently, behavior of the composite section will be affected by this different age. The proper calculation of the resulting deflection is very complex. In this example, the deflection due to differing age concrete is approximated as one-half of the dead load deflection of the beam due to the slab dead load. Readers are cautioned that this procedure results in only a rough estimate. Half of the dead load deflection is

$$\Delta_{ds} = 0.50\,(\Delta_i)_1 = (0.50)\,(0.358) = 0.179 \text{ in. (rough estimate)}$$

Term (9) — Using the alternative method

$$(\Delta_i)_\ell = \frac{K\,(5/48)\,M_\ell \ell^2}{(E_c)_2\,(I_c)_{cr}} = \frac{(1)\,(5/48)\,(50.7)\,(26)^2\,(12)^3}{(3830)\,(8295)} = 0.194 \text{ in.}$$

Term (10) — $k_r = 0.85$ (neglecting the effect of any compression steel in slab)

$$(\Delta_{cp})_\ell = k_r C_u \left[0.20\,(\Delta_i)_\ell\right]$$

$$= (0.85)\,(1.60)\,(0.20 \times 0.194) = 0.053 \text{ in.}$$

In Eq. (19), $\Delta_u = 0.084 + 0.055 + 0.023 + 0.069 + 0.047 + 0.358 + 0.166 + 0.179 + 0.194 + 0.053$
$= 1.23$ in.

Checking Eq. (20) (same solution),

$$\Delta_u = \left(1.65 + 0.71\frac{I_2}{I_c}\right)(\Delta_i)_2 + \left(0.36 + 0.64\frac{I_2}{I_c}\right)\Delta_{sh} +$$

$$\left(1.05 + 1.21\frac{I_2}{I_c}\right)(\Delta_i)_1 + (\Delta_i)_\ell + (\Delta_{cp})_\ell$$

$= (1.65 + 0.71 \times 0.383)(0.084) + (0.36 + 0.64 \times 0.383)(0.192)$
$+ (1.50 + 1.21 \times 0.383)(0.358) + 0.194 + 0.053$
$=$ 1.23 in. (same as above)Assuming nonstructural elements are installed after the composite slab has hardened,

$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell = \text{Terms (3) + (5) + (7) + (8) + (9) + (10)}$$

$$= 0.023 + 0.047 + 0.166 + 0.179 + 0.194 + 0.053 = 0.66 \text{ in.}$$

Comparisons with the allowable deflections in Table 9.5(b) are shown at the end of Design Example 10.7.

Example 10.7—Shored Nonprestressed Composite Beam

Required: Analysis of short-term and ultimate long-term deflections, to show the beneficial effect of shoring in reducing deflections.

Data: Same as in Example 10.6, except that shored construction is used.

Calculations and Discussion	Code Reference
1. Effective moments of inertia for composite section, using Eq. (9-8):	
$M_{cr} = f_r I_g / y_t = (474)(26{,}470)/(16.29)(12{,}000) = 64.2$ ft-kips	*Eq. (9-9)*
$M_{cr}/(M_1 + M_2) = [64.2/(40.6 + 21.1)] = 1.04 > 1$	
Hence $(I_e)_{1+2} = I_g = 26{,}470$ in.^{4}	
In Term (5), Eq. (17)—Composite Section,	
$[M_{cr}/(M_1 + M_2 + M_\ell)]^3 = [64.2/(40.6 + 21.1 + 50.7)]^3 = 0.186$	
$(I_e)_{d+\ell} = (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3] I_{cr} \le I_g$	*Eq. (9-8)*
$= (0.186)(26{,}470) + (1 - 0.186)(8295) = 11{,}675$ in.^{4}	

versus the alternative method of Example 8.6 where $I_e = (I_c)_{cr} = 8295$ in.^{4} was used with the live load moment directly.

2. Deflections, using Eqs. (17) and (18):

Term (1) — $$(\Delta_i)_{1+2} = \frac{K(5/48)(M_1 + M_2)\ell^2}{(E_c)_2 (I_e)_{1+2}}$$

$$= \frac{(1)(5/48)(40.6 + 21.1)(26)^2(12)^3}{(3830)(26{,}470)} = 0.074 \text{ in.}$$

Term (2) —Creep deflection due to total dead load of beam and slab. The value of C_u for the beam is taken to be 1.60. Consider the value of C_u for the slab to be slightly higher. For shores removed at 10 days, it may be shown by comparison of the correction factors, K^c_{to} for 10 and 20 day load applications (Section 2.3.4, ACI 435, Ref. 10.4) that the ultimate creep coefficient for the slab is approximately 1.74. k_r is conservatively assumed to have a value of 0.85.

The average creep coefficient for the composite section is:

$\text{Avg.}C_u = \frac{1}{2}(1.60 + 1.74) = 1.67$

$1.67 k_r (\Delta_i)_{1+2} = (1.67)(0.85)(0.074) = 0.105$ in.

Term (3) — Shrinkage deflection of the precast beam after shores are removed. As indicated in Term 4 of Example 8.6, the fraction of shrinkage of the precast beam at 2 months is 0.36. The shores are assumed to be removed about 10 days after the 2-month point. Therefore, consider the remaining fraction of shrinkage is 1-0.36 = 0.64. Recall that the ultimate shrinkage, $(\varepsilon_{sh})_u = 400 \times 10^{-6}$. Utilize the result found for Δ_{sh} in Term (4) of Example 10.6:

Remaining $(e_{sh}) = (0.64)(400 \times 10^{-6}) = 256 \times 10^{-6}$

$$\Delta_{sh}\frac{I_2}{I_c} = (256/400)\,(0.192)\,(0.383) = 0.047 \text{ in.}$$

Term (4) — Deflection due to differences in shrinkage and creep in the beam and slab. This is a complex issue. For this example, assume that the magnitude of this component is approximated by the initial dead load deflection of the composite section.

$$\Delta_{ds} = (\Delta_i)_{1+2} = 0.074 \text{ in. (rough estimate)}$$

Term (5) — $$(\Delta_i)_\ell = \frac{K\,(5/48)\,(M_1 + M_2 + M_\ell)\ell^2}{(E_c)_2\,(I_e)_{d+\ell}} - (\Delta_i)_{1+2}$$

$$= \frac{(1)\,(5/48)\,(40.6 + 21.1 + 50.7)\,(26)^2\,(12)^3}{(3830)\,(11{,}675)} - 0.074 \text{ in.} = 0.232 \text{ in.}$$

Term (6) — $k_r = 0.85$ (neglecting the effect of any compression steel in slab),

$$(\Delta_{cp})_\ell = k_r C_u\,[0.20\,(\Delta_i)_\ell] = (0.85)\,(1.60)\,(0.20 \times 0.232) = 0.063 \text{ in.}$$

In Eq. (17), $\Delta_u = 0.074 + 0.105 + 0.047 + 0.074 + 0.232 + 0.063 = 0.60$ in.

versus 1.23 in. with unshored construction.

This shows the beneficial effect of shoring in reducing the total deflection.

Checking by Eq. (18) (same solution),

$$\Delta_u = 3.42\,(\Delta_i)_{1+2} + \Delta_{sh}\frac{I_2}{I_c} + (\Delta_i)_\ell + (\Delta_{cp})_\ell$$

$$= (3.42)\,(0.074) + 0.046 + 0.232 + 0.063 = 0.60 \text{ in. (same as above)}$$

Assuming that nonstructural elements are installed after shores are removed,

$$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell = \Delta_u - (\Delta_i)_{1+2} = 0.60 - 0.07 = 0.53 \text{ in.}$$

Comparison of Results of Examples 10.6 and 10.7

The computed deflections of $(\Delta_i)_\ell = 0.19$ in. in Example 10.6 and 0.23 in. in Example 10.7; and $\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell = 0.66$ in. in Example 10.6 and 0.53 in. in Example 10.7 are compared with the allowable deflections in Table 9.5(b) as follows:

Flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$(\Delta_i)_\ell \leq \ell/180 = 312/180 = 1.73$ in. O.K.

Floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections—

$(\Delta_i)_\ell \leq \ell/360 = 312/360 = 0.87$ in. O.K.

Roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections (very stringent limitation)—

$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \leq \ell/480 = 312/480 = 0.65$ in.

Note that the long term deflection occurring after attachment of non-structural elements (0.66 in) exceeds the L/480 limit. It actually meets L/473. Since the L/480 limit only applies in case of *nonstructural elements likely to be damaged by large deflections*, the particular use of the beam would have to be considered in order to make a judgment on the acceptability of the computed deflections. Refer to the footnotes following Table 9.5(b) of ACI 318

Roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections—

$\Delta_{cp} + \Delta_{sh} + (\Delta_i)_\ell \leq \ell/240 = 312/240 = 1.30$ in. O.K.

11

Design for Slenderness Effects

UPDATE FOR THE '05 CODE

Section 10.13.6 was modified to clarify the load factors to be used for investigating the strength and stability of the structure as a whole under gravity loads in 10.13.6(a) and (b).

GENERAL CONSIDERATIONS

Design of columns consists essentially of selecting an adequate column cross-section with reinforcement to support required combinations of factored axial loads P_u and factored (primary) moments M_u, including consideration of column slenderness (secondary moments).

Column slenderness is expressed in terms of its slenderness ratio $k\ell_u/r$, where k is an effective length factor (dependent on rotational and lateral restraints at the ends of the column), ℓ_u is the unsupported column length, and r is the radius of gyration of the column cross-section. In general, a column is slender if its applicable cross-sectional dimension is small in comparison to its length.

For design purposes, the term "short column" is used to denote a column that has a strength equal to that computed for its cross-section, using the forces and moments obtained from an analysis for combined bending and axial load. A "slender column" is defined as a column whose strength is reduced by second-order deformations (secondary moments). By these definitions, a column with a given slenderness ratio may be considered a short column for design under one set of restraints, and a long column under another set. With the use of higher strength concrete and reinforcement, and with more accurate analysis and design methods, it is possible to design smaller cross-sections, resulting in members that are more slender. The need for reliable and rational design procedures for slender columns thus becomes a more important consideration in column design.

A short column may fail due to a combination of moment and axial load that exceeds the strength of the cross-section. This type of a failure is known as "material failure." As an illustration, consider the column shown in Fig. 11-1. Due to loading, the column has a deflection Δ which will cause an additional (secondary) moment in the column. From the free body diagram, it can be seen that the maximum moment in the column occurs at section A-A, and is equal to the applied moment plus the moment due to member deflection, which is $M = P(e + \Delta)$.

Failure of a short column can occur at any point along the strength interaction curve, depending on the combination of applied moment and axial load. As discussed above, some deflection will occur and a "material failure" will result when a particular combination of load P and moment $M = P(e + \Delta)$ intersects the strength interaction curve.

If a column is very slender, it may reach a deflection due to axial load P and a moment Pe such that deflections will increase indefinitely with an increase in the load P. This type of failure is known as a "stability failure," as shown on the strength interaction curve.

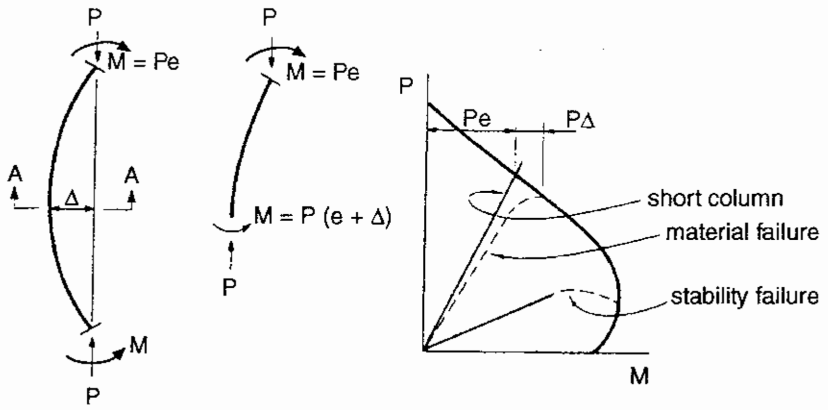


Figure 11-1 Strength Interaction for Slender Columns

The basic concept on the behavior of straight, concentrically loaded, slender columns was originally developed by Euler more than 200 years ago. It states that a member will fail by buckling at the critical load $P_c = \pi^2 EI/(\ell_e)^2$, where EI is the flexural stiffness of the member cross-section, and ℓ_e is the effective length, which is equal to $k\ell_u$. For a "stocky" short column, the value of the buckling load will exceed the direct crushing strength (corresponding to material failure). In members that are more slender (i.e., members with larger $k\ell_u/r$ values), failure may occur by buckling (stability failure), with the buckling load decreasing with increasing slenderness (see Fig. 11-2).

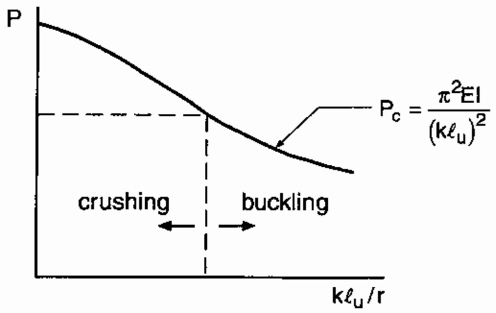


Figure 11-2 Failure Load as a Function of Column Slenderness

As shown above, it is possible to depict slenderness effects and amplified moments on a typical strength interaction curve. Hence, a "family" of strength interaction diagrams for slender columns with varying slenderness ratios can be developed, as shown in Fig. 11-3. The strength interaction diagram for $k\ell_u/r = 0$ corresponds to the combinations of moment and axial load where strength is not affected by member slenderness (short column strength).

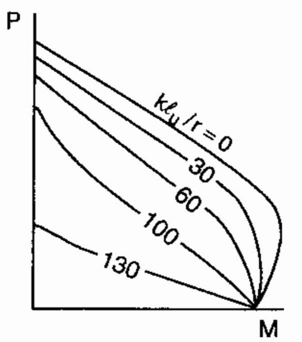
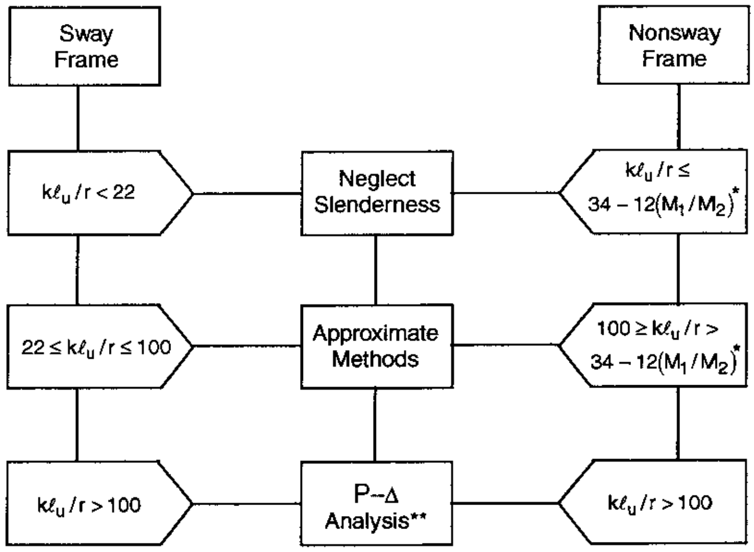


Figure 11-3 Strength Interaction Diagrams for Slender Columns

CONSIDERATION OF SLENDERNESS EFFECTS

Slenderness limits are prescribed for both nonsway and sway frames, including design methods permitted for each slenderness range. Lower-bound slenderness limits are given, below which secondary moments may be disregarded and only axial load and primary moment need be considered to select a column cross-section and reinforcement (short column design). It should be noted that for ordinary beam and column sizes and typical story heights of concrete framing systems, effects of slenderness may be neglected for more than 90 percent of columns in nonsway frames and around 40 percent of columns in sway frames. For moderate slenderness ratios, an approximate analysis of slenderness effects based on a moment magnifier (see 10.12 and 10.13) is permitted. For columns with high slenderness ratios, a more exact second-order analysis is required (see 10.11.5), taking into account material nonlinearity and cracking, as well as the effects of member curvature and lateral drift, duration of the loads, shrinkage and creep, and interaction with the supporting foundation. No upper limits for column slenderness are prescribed. The slenderness ratio limits in 10.12.2 for nonsway frames and 10.13.2 for sway frames, and design methods permitted for consideration of column slenderness, are summarized in Fig. 11-4.



* $34 - 12(M_1/M_2) \le 40$

** Permitted for any slenderness ratio

Figure 11-4 Consideration of Column Slenderness

10.10 SLENDERNESS EFFECTS IN COMPRESSION MEMBERS

10.10.1 Second-Order Frame Analysis

The code encourages the use of a second-order frame analysis or P-Δ analysis for consideration of slenderness effects in compression members. Generally, the results of a second-order analysis give more realistic values of the moments than those obtained from an approximate analysis by 10.12 or 10.13. For sway frames, the use of second-order analyses will generally result in a more economical design. Procedures for carrying out a second-order analysis are given in Commentary Refs. 10.31-10.32. The reader is referred to R10.10.1, which discusses minimum requirements for an adequate second-order analysis under 10.10.1.

If more exact analyses are not feasible or practical, 10.10.2 permits an approximate moment magnifier method to account for column slenderness. Note, however, that for all compression members with a column slenderness ratio ($k\ell_u/r$) greater than 100 (see Fig. 11-4), a more exact analysis as defined in 10.10.1 must be used for consideration of slenderness effects.

10.11 APPROXIMATE EVALUATION OF SLENDERNESS EFFECTS

The moment magnification factor δ is used to magnify the primary moments to account for increased moments due to member curvature and lateral drift. The moment magnifier δ is a function of the ratio of the applied axial load to the critical or buckling load of the column, the ratio of the applied moments at the ends of the column, and the deflected shape of the column.

10.11.1 Section Properties for Frame Analysis

According to 10.11.1, the factored axial loads P_u, the factored moments at the column ends M_1 and M_2, and the relative lateral story deflections Δ_o shall be computed using an elastic first-order frame analysis taking into account cracked regions along the length of the members. It is usually not economically feasible to perform such calculations even for small structures. Thus, the section properties given in 10.11.1 and summarized in Table 11-1 may be used in the analysis to account for cracking. The values of E, I, and A have been chosen from the results of frame tests and analyses as outlined in code Reference 10.33. It is important to note that for service load analysis of the structure, it is satisfactory to multiply the moments of inertia given in Table 11-1 by 1/0.70 = 1.43 (R10.11.1). Also, the moments of inertia must be divided by $(1 + \beta_d)$ in the case when sustained lateral loads act on the structure (for example, lateral loads resulting from earth pressure) or when the gravity load stability check made in accordance with 10.13.6 is performed.

Table 11-1 Section Properties for Frame Analysis

	Modulus of Elasticity	Moment of Inertia†	Area
Beams		$0.35\ I_g$	
Columns		$0.70\ I_g$	
Walls - uncracked	E_c from 8.5.1	$0.70\ I_g$	$1.0 A_g$
Walls - cracked		$0.35\ I_g$	
Flat plates and flat slabs		$0.25\ I_g$	

†Divide by $(1+\beta_d)$ when sustained lateral loads act or for stability checks made in accordance with 10.13.6. For service load analyses, multiply by 1/0.70 = 1.43.

10.11.2 Radius of Gyration

In general, the radius of gyration, r, is $\sqrt{I_g/A_g}$. In particular, r may be taken as 0.30 times the dimension in the direction of analysis for a rectangular section and 0.25 times the diameter of a circular section, as shown in Fig. 11-5.

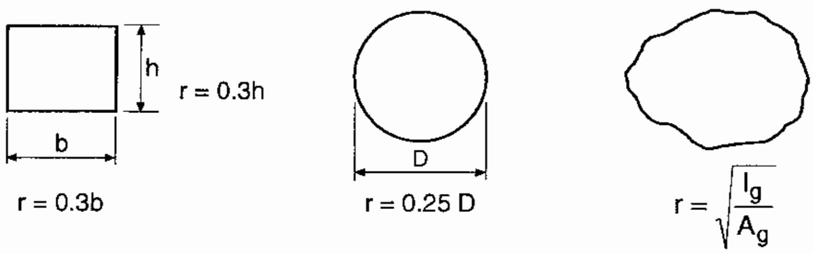


Figure 11-5 Radius of Gyration, r

10.11.3, 10.12.1 Unsupported and Effective Lengths of Compression Members

The unsupported length ℓ_u of a column, defined in 10.11.3, is the clear distance between lateral supports as shown in Fig. 11-6. Note that the length ℓ_u may be different for buckling about each of the principal axes of the column cross-section. The basic Euler equation for critical buckling load can be expressed as $P_c = \pi^2 EI/(\ell_e)^2$, where ℓ_e is the effective length $k\ell_u$. The basic equations for the design of slender columns were derived for hinged ends, and thus, must be modified to account for the effects of end restraint. Effective column length $k\ell_u$, as contrasted to actual unbraced length ℓ_u, is the term used in estimating slender column strength, and considers end restraints as well as nonsway and sway conditions.

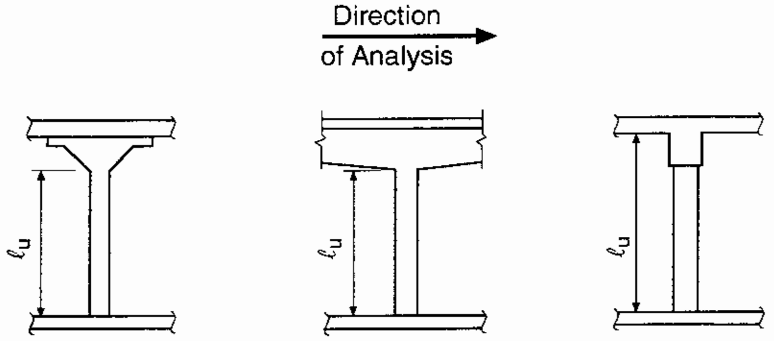


Figure 11-6 Unsupported Length, ℓ_u

At the critical load defined by the Euler equation, an originally straight member buckles into a half-sine wave as shown in Fig. 11-7(a). In this configuration, an additional moment P-Δ acts at every section, where Δ is the lateral deflection at the specific location under consideration along the length of the member. This deflection continues to increase until the bending stress caused by the increasing moment (P-Δ), plus the original compression stress caused by the applied loading, exceeds the compressive strength of concrete and the member fails. The effective length ℓ_e (= $k\ell_u$) is the length between pinned ends, between zero moments or between inflection points. For the pinned condition illustrated in Fig. 11-7(a), the effective length is equal to the unsupported length ℓ_u. If the member is fixed against rotation at both ends, it will buckle in the shape depicted in Fig. 11-7(b); inflection points will occur at the locations shown, and the effective length ℓ_e will be one-half of the unsupported length. The critical buckling load P_c for the fixed-end condition is four times that for a pin-end condition. Rarely are columns in actual structures either hinged or fixed; they are partially restrained against rotation by members framing into the column, and thus the effective length is between $\ell_u/2$ and ℓ_u, as shown in Fig. 11-7(c) as long as the lateral displacement of one end of the column with respect to the other end is prevented. The actual value of the effective length depends on the rigidity of the members framing into the top and bottom ends of the column.

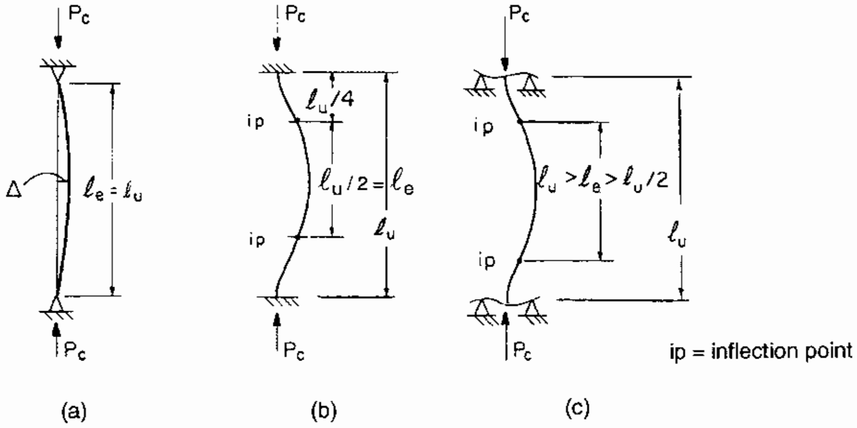


Figure 11-7 Effective Length, ℓ_e (Nonsway Condition)

A column that is fixed at one end and entirely free at the other end (cantilever) will buckle as shown in Fig. 11-8(a). The upper end will deflect laterally relative to the lower end; this is known as sidesway. The deflected shape of such a member is similar to one-half of the sinusoidal deflected shape of the pin-ended member illustrated in Fig. 11-7(a). The effective length is equal to twice the actual length. If the column is fixed against rotation at both ends but one end can move laterally with respect to the other, it will buckle as shown in Fig. 11-8(b). The effective length ℓ_e will be equal to the actual length ℓ_u, with an inflection point (ip) occurring as shown. The buckling load of the column in Fig. 11-8(b), where sidesway is not prevented, is one-quarter that of the column in Fig. 11-7(b), where sidesway is prevented. As noted above, the ends of columns are rarely either completely hinged or completely fixed, but rather are partially restrained against rotation by members framing into the ends of the columns. Thus, the effective length will vary between ℓ_u and infinity, as shown in Fig. 11-8(c). If restraining members (beams or slab) are very rigid as compared to the column, the buckling in Fig. 11-8(b) is approached. If, however, the restraining members are quite flexible, a hinged condition is approached at both ends and the column(s), and possibly the structure as a whole, approaches instability. In general, the effective length ℓ_e depends on the degree of rotational restraint at the ends of the column, in this case $\ell_u < \ell_e < \infty$.

In typical reinforced concrete structures, the designer is rarely concerned with single members, but rather with rigid framing systems consisting of beam-column and slab-column assemblies. The buckling behavior of a frame that is not braced against sidesway can be illustrated by the simple portal frame shown in Fig. 11-9. Without lateral restraint at the upper end, the entire (unbraced) frame is free to move sideways. The bottom end may be pinned or partially restrained against rotation.

In summary, the following comments can be made:

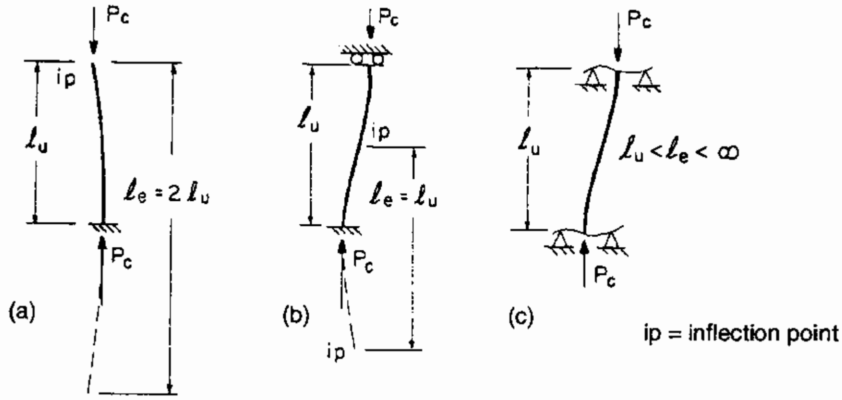


Figure 11-8 Effective Length, ℓ_e (Sway Condition)

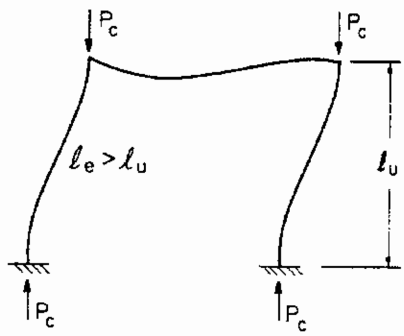


Figure 11-9 Rigid Frame (Sway Condition)

1. For compression members in a nonsway frame, the effective length ℓ_e falls between $\ell_u/2$ and ℓ_u, where ℓ_u is the actual unsupported length of the column.

2. For compression members in a sway frame, the effective length ℓ_e is always greater than the actual length of the column ℓ_u, and may be $2\ell_u$ and higher. In this case, a value of k less than 1.2 normally would not be realistic.

3. Use of the alignment charts shown in Figs. 11-10 and 11-11 (also given in Fig. R10.12.1) allows graphical determination of the effective length factors for compression members in nonsway and sway frames, respectively. If both ends of a column in a nonsway frame have minimal rotational stiffness, or approach $\psi = \infty$, then k = 1.0. If both ends have or approach full fixity, $\psi = 0$, and k = 0.5. If both ends of a column in a sway frame have minimal rotational stiffness, or approach $\psi = \infty$, then $k = \infty$. If both ends have or approach full fixity, $\psi = 0$, then k = 1.0.

An alternative method for computing the effective length factors for compression members in nonsway and sway frames is contained in R10.12.1. For compression members in a nonsway frame, an upper bound to the effective length factor may be taken as the smaller of the values given by the following two expressions, which are given in the 1972 British Standard Code of Practice (ACI Refs. 10.38 and 10.39):

$$k = 0.7 + 0.05(\psi_A + \psi_B) \le 1.0$$

$$k = 0.85 + 0.05\,\psi_{min} \le 1.0$$

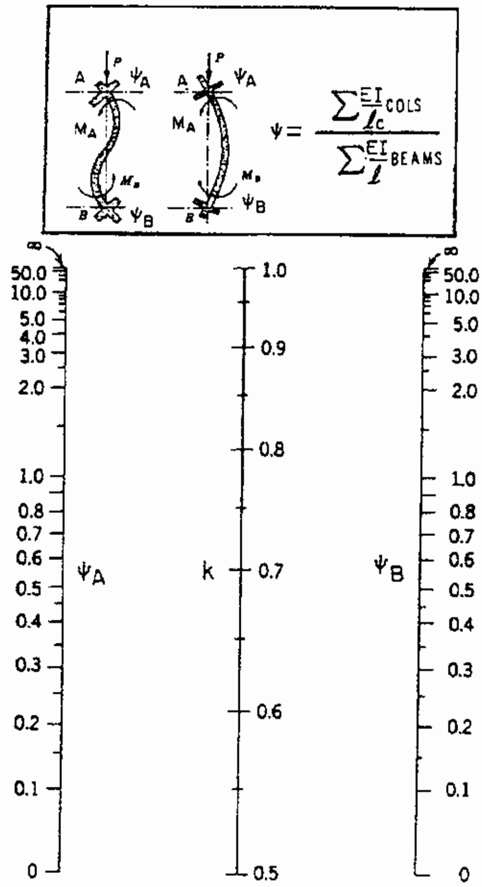


Figure 11-10 Effective Length Factors for Compression Members in a Nonsway Frame

where ψ_A and ψ_B are the values of ψ at the ends of the column and ψ_{min} is the smaller of the two values.

For compression members in a sway frame restrained at both ends, the effective length factor may be taken as (ACI Ref. 10.25):

$$\text{For } \psi_m < 2, \; k = \frac{20 - \psi_m}{20}\sqrt{1 + \psi_m}$$

$$\text{For } \psi_m \geq 2, \; k = 0.9\sqrt{1 + \psi_m}$$

where ψ_m is the average of the ψ values at the two ends of the column.

For compression members in a sway frame hinged at one end, the effective length factor may be taken as (ACI Refs. 10.38 and 10.39):

$$k = 2.0 + 0.3\psi$$

where ψ is the column-to-beam stiffness ratio at the restrained end.

In determining the effective length factor k from Figs. 11-10 and 11-11, or from the Commentary equations, the rigidity (EI) of the beams (or slabs) and columns shall be calculated based on the values given in 10.11.1.

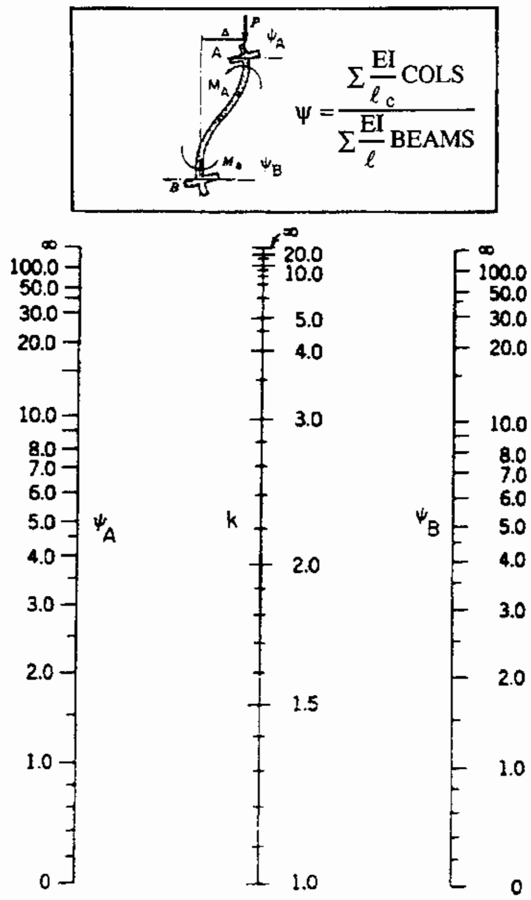


Figure 11-11 Effective Length Factors for Compression Members in a Sway Frame

10.11.4 Nonsway Versus Sway Frames

In actual structures, there is rarely a completely nonsway or sway condition. If it is not readily apparent by inspection, 10.11.4.1 and 10.11.4.2 give two possible ways of determining if a frame is nonsway or not. According to 10.11.4.1, a column in a structure can be considered nonsway if the column end moments due to second-order effects do not exceed 5 percent of the first-order end moments. According to 10.11.4.2, it is also permitted to assume a story within a structure is nonsway if:

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us} \ell_c} \leq 0.05 \qquad \textit{Eq. (10-6)}$$

where

Q = stability index for a story
ΣP_u = total factored vertical load in the story corresponding to the lateral loading case for which ΣP_u is greatest (R10.11.4)
V_{us} = factored horizontal shear in the story
Δ_o = first-order relative deflection between the top and bottom of the story due to V_u
ℓ_c = column length, measured from center-to-center of the joints in the frame

Note that Eq. (10-6) is not applicable when $V_u = 0$.

10.11.6 Moment Magnifier δ for Biaxial Bending

When biaxial bending occurs in a column, the computed moments about each of the principal axes must be magnified. The magnification factors δ are computed considering the buckling load P_c about each axis separately, based on the appropriate effective lengths and the related stiffness ratios of columns to beams in each direction. Thus, different buckling capacities about the two axes are reflected in different magnification factors. The moments about each of the two axes are magnified separately, and the cross-section is then proportioned for an axial load P_u and magnified biaxial moments.

10.12.2, 10.13.2 Consideration of Slenderness Effects

For compression members in a nonsway frame, effects of slenderness may be neglected when $k\ell_u/r$ is less than or equal to 34 - 12 (M_1/M_2), where M_2 is the larger end moment and M_1 is the smaller end moment. The ratio M_1/M_2 is positive if the column is bent in single curvature, negative if bent in double curvature. Note that M_1 and M_2 are factored end moments obtained by an elastic frame analysis and that the term [34-12M_1/M_2] shall not be taken greater than 40. For compression members in a sway frame, effects of slenderness may be neglected when $k\ell_u/r$ is less than 22 (10.13.2). The moment magnifier method may be used for columns with slenderness ratios exceeding these lower limits.

The upper slenderness limit for columns that may be designed by the approximate moment magnifier method is $k\ell_u/r$ equal to 100 (10.11.5). When $k\ell_u/r$ is greater than 100, an analysis as defined in 10.10.1 must be used, taking into account the influence of axial loads and variable moment of inertia on member stiffness and fixed-end moments, the effect of deflections on the moments and forces, and the effects of duration of loading (sustained load effects). Criteria for consideration of column slenderness are summarized in Fig. 11-4.

The lower slenderness ratio limits will allow a large number of columns to be exempt from slenderness consideration. Considering the slenderness ratio $k\ell_u/r$ in terms of ℓ_u/h for rectangular columns, the effects of slenderness may be neglected in design when ℓ_u/h is less than 10 for compression members in a nonsway frame and with zero moments at both ends. This lower limit increases to 15 for a column in double curvature with equal end moments and a column-to-beam stiffness ratio equal to one at each end. For columns with minimal or zero restraint at both ends, a value of k equal to 1.0 should be used. For stocky columns restrained by flat slab floors, k ranges from about 0.95 to 1.0 and can be conservatively estimated as 1.0. For columns in beam-column frames, k ranges from about 0.75 to 0.90, and can be conservatively estimated as 0.90. If the initial computation of the slenderness ratio based on estimated values of k indicates that effects of slenderness must be considered in the design, a more accurate value of k should be calculated and slenderness re-evaluated. For a compression member in a sway frame with a column-to-beam stiffness ratio equal to 1.0 at both ends, effects of slenderness may be neglected when ℓ_u/h is less than 5. This value reduces to 3 if the beam stiffness is reduced to one-fifth of the column stiffness at each end of the column. Thus, beam stiffnesses at the top and bottom of a column of a high-rise structure where sidesway is not prevented by structural walls or other means will have a significant effect on the degree of slenderness of the column.

The upper limit on the slenderness ratio of $k\ell_u/r$ equal to 100 corresponds to an ℓ_u/h equal to 30 for a compression member in a nonsway frame with zero restraint at both ends. This ℓ_u/h limit increases to 39 for a column-to-beam stiffness ratio of 1.0 at each end.

10.12.3 Moment Magnification—Nonsway Frames

The approximate slender column design equations contained in 10.12.3 for nonsway frames are based on the concept of a moment magnifier δ_{ns} which amplifies the larger factored end moment M_2 on a compression member. The column is then designed for the factored axial load P_u and the amplified moment M_c where M_c is given by:

$$M_c = \delta_{ns} M_2 \qquad \textit{Eq. (10-8)}$$

where

$$\delta_{ns} = \frac{C_m}{1 - \dfrac{P_u}{0.75P_c}} \geq 1.0 \qquad \textit{Eq. (10-9)}$$

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2} \qquad \textit{Eq. (10-10)}$$

The critical load P_c is computed for a nonsway condition using an effective length factor k of 1.0 or less. When k is determined from the alignment charts or the equations in R10.12, the values of E and I from 10.11.1 must be used in the computations of ψ_A and ψ_B. Note that the 0.75 factor in Eq. (10-9) is a stiffness reduction factor (see R10.12.3).

In defining the critical column load P_c, the difficult problem is the choice of a stiffness parameter EI which reasonably approximates the stiffness variations due to cracking, creep, and the nonlinearity of the concrete stress-strain curve. In lieu of a more exact analysis, EI shall be taken as:

$$EI = \frac{(0.2E_cI_g + E_sI_{se})}{1 + \beta_d} \qquad \textit{Eq. (10-11)}$$

or

$$EI = \frac{0.4E_cI_g}{1 + \beta_d} \qquad \textit{Eq. (10-12)}$$

The second of these two equations is a simplified approximation to the first. Both equations approximate the lower limits of EI for practical cross-sections and, thus, are conservative. The approximate nature of the EI equations is shown in Fig. 11-12 where they are compared with values derived from moment-curvature diagrams for the case when there is no sustained load ($\beta_d = 0$).

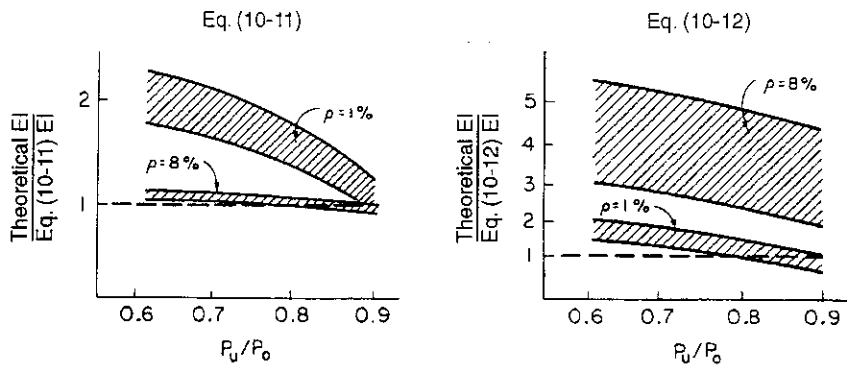


Figure 11-12 Comparison of Equations for EI with EI Values from Moment-Curvature Diagrams

Equation (10-11) represents the lower limit of the practical range of stiffness values. This is especially true for heavily reinforced columns. As noted above, Eq. (10-12) is simpler to use but greatly underestimates the effect of reinforcement in heavily reinforced columns (see Fig. 11-12).

Both EI equations were derived for small e/h values and high P_u/P_o values, where the effect of axial load is most pronounced. The term P_o is the nominal axial load strength at zero eccentricity.

For reinforced concrete columns subjected to sustained loads, creep of concrete transfers some of the load from the concrete to the steel, thus increasing steel stresses. For lightly reinforced columns, this load transfer may cause compression steel to yield prematurely, resulting in a loss in the effective value of EI. This is taken into account by dividing EI by $(1 + \beta_d)$. For nonsway frames, β_d is defined as follows (see 10.11.1):

$$\beta_d = \frac{\text{Maximum factored axial sustained load}}{\text{Maximum factored axial load associated with the same load combination}}$$

For composite columns in which a structural steel shape makes up a large percentage of the total column cross-section, load transfer due to creep is not significant. Accordingly, only the EI of the concrete portion should be reduced by $(1 + \beta_d)$ to account for sustained load effects.

The term C_m is an equivalent moment correction factor. For members without transverse loads between supports, C_m is (10.12.3.1):

$$C_m = 0.6 + 0.4\left(\frac{M_1}{M_2}\right) \geq 0.4 \qquad \textit{Eq. (10-13)}$$

For members with transverse loads between supports, it is possible that the maximum moment will occur at a section away from the ends of a member. In this case, the largest calculated moment occurring anywhere along the length of the member should be magnified by δ_{ns}, and C_m must be taken as 1.0. Figure 11-13 shows some values of C_m, which are a function of the end moments.

If the computed column moment M_2 in Eq. (10-8) is small or zero, design of a nonsway column must be based on the minimum moment $M_{2,min}$ (10.12.3.2):

$$M_{2,min} = P_u\,(0.6 + 0.03h) \qquad \textit{Eq. (10-14)}$$

For members where $M_{2,min} > M_2$, the value of C_m shall either be taken equal to 1.0, or shall be computed by Eq. (10-13) using the ratio of the actual computed end moments M_1 and M_2.

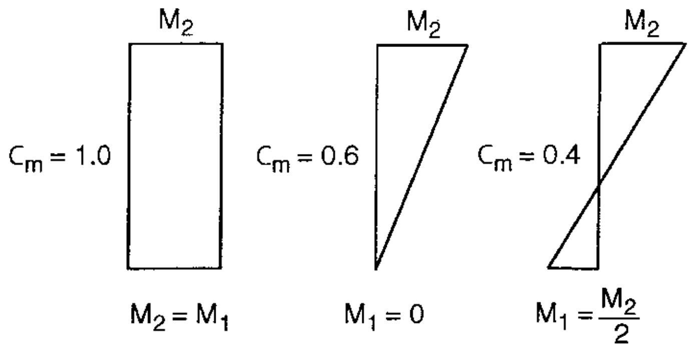


Figure 11-13 Moment Factor C_m

10.13.3 Moment Magnification—Sway Frames

The design of sway frames for slenderness consists essentially of three steps:

1. The magnified sway moments $\delta_s M_s$ are computed in one of three ways:
   a. A second-order elastic frame analysis (10.13.4.1)
   b. An approximate second-order analysis (10.13.4.2)
   c. An approximate magnifier method given in earlier ACI codes (10.13.4.3)
2. The magnified sway moments $\delta_s M_s$ are added to the unmagnified nonsway moments M_{ns} at each end of the column (10.13.3):

   $$M_1 = M_{1ns} + \delta_s M_{1s} \qquad \textit{Eq. (10-15)}$$

   $$M_2 = M_{2ns} + \delta_s M_{2s} \qquad \textit{Eq. (10-16)}$$

   The nonsway moments M_{1ns} and M_{2ns} are computed using a first-order elastic analysis.
3. If the column is slender and subjected to high axial loads, it must be checked to see whether moments at points between the column ends are larger than those at the ends. According to 10.13.5, this check is performed using the nonsway magnifier δ_{ns} with P_c computed assuming k = 1.0 or less.

10.13.4 Calculation of $\delta_s M_s$

As noted above, there are three different ways to compute the magnified sway moments $\delta_s M_s$. If a second-order elastic analysis is used to compute $\delta_s M_s$, the deflections must be representative of the stage immediately prior to the ultimate load. Thus, the values of EI given in 10.11.1 must be used in the second-order analysis. Note that I must be divided by $(1 + \beta_d)$ where for sway frames, β_d is defined as follows (see 10.11.1):

$$\beta_d = \frac{\text{Maximum factored sustained shear within a story}}{\text{Maximum factored shear in that story}}$$

For wind or earthquake loads, $\beta_d = 0$. An example of a non-zero β_d may occur when members are subjected to earth pressure.

Section 10.13.4.2 allows an approximate second-order analysis to determine $\delta_s M_s$. In this case, the solution of the infinite series that represents the iterative P-Δ analysis for second-order moments is given as follows:

$$\delta_s M_s = \frac{M_s}{1 - Q} \geq M_s \qquad \text{Eq. (10-17)}$$

where

Q = stability index for a story

$$= \frac{\Sigma P_u \Delta_o}{V_{us} \ell_c} \qquad \text{Eq. (10-6)}$$

Note that Eq. (10-17) closely predicts the second-order moments in a sway frame until δ_s exceeds 1.5. For the case when $\delta_s > 1.5$, $\delta_s M_s$ must be computed using 10.13.4.1 or 10.13.4.3.

The code also allows $\delta_s M_s$ to be determined using the magnified moment procedure that was given in previous ACI codes (10.13.4.3):

$$\delta_s M_s = \frac{M_s}{1 - \dfrac{\Sigma P_u}{0.75 \Sigma P_c}} \geq M_s \qquad \text{Eq. (10-18)}$$

where

ΣP_u = summation of all the factored vertical loads in a story

ΣP_c = summation of the critical buckling loads for all sway-resisting columns in a story

It is important to note that the moment magnification in the columns farthest from the center of twist in a building subjected to significant torsional displacement may be underestimated by the moment magnifier procedure. A three-dimensional second-order analysis should be considered in such cases.

10.13.5 Location of Maximum Moment

When the unmagnified nonsway moments at the ends of the column are added to the magnified sway moments at the same points, one of the resulting total end moments is usually the maximum moment in the column. However, for slender columns with high axial loads, the maximum moment may occur between the ends of the column. A simple way of determining if this situation occurs or not is given in 10.13.5: if an individual compression member has

$$\frac{\ell_u}{r} > \frac{35}{\sqrt{\dfrac{P_u}{f'_c A_g}}} \qquad \text{Eq. (10-19)}$$

the maximum moment will occur at a point between the ends of the column. In this case, M_2, which is defined in Eq. (10-16), must be magnified by the nonsway moment magnifier given in Eq. (10-9). The column is then designed for the factored axial load P_u and the moment M_c, where M_c is computed from the following:

$$M_c = \delta_{ns} M_2 = \delta_{ns}(M_{2ns} + \delta_s M_{2s}) \qquad \text{Eq. (10-8)}$$

Where:

$$\delta_{ns} = \frac{C_m}{1 - \dfrac{P_u}{0.75P_c}}$$

Note that to calculate δ_{ns} k is determined according to 10.12.1 and $\delta_{ns} \geq 1.0$, and M_1 and M_2 given by Eqs. (10-15), (10-16) are used to calculate C_m according to Eq. (10-13).

10.13.6 Structural Stability Under Gravity Loads

For sway frames, the possibility of sidesway instability of the structure as a whole under factored gravity loads must be investigated. This is checked in three different ways, depending on the method that is used in determining $\delta_s M_s$:

1. When $\delta_s M_s$ is computed by a second-order analysis (10.13.4.1), the following expression must be satisfied:

$$\frac{\text{Second-order lateral deflections}}{\text{First-order lateral deflections}} \leq 2.5$$

   Note that these deflections are based on the applied loading of $1.2P_D$ and $1.6P_L$ plus factored lateral load. The frame should be analyzed twice for this set of applied loads: the first analysis should be a first-order analysis and the second should be a second-order analysis. The lateral load may be the actual lateral loads used in design or it may be a single lateral load applied to the top of the frame. In any case, the lateral load(s) should be large enough to give deflections that can be compared accurately.

2. When $\delta_s M_s$ is computed by the approximate second-order analysis (10.13.4.2), then

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us} \ell_c} \leq 0.60$$

   where the value of Q is evaluated using $1.2P_D$ and $1.6P_L$. Note that the above expression is equivalent to $\delta_s = 2.5$. The values of V_{us} and Δ_o may be determined using the actual or any arbitrary set of lateral loads. The above stability check is satisfied if the value of Q computed in 10.11.4.2 is less than or equal to 0.2.

3. When $\delta_s M_s$ is computed using the expressions from previous ACI codes (10.13.4.3), the stability check is satisfied when

$$0 < \delta_s \leq 2.5$$

   In this case, ΣP_u and ΣP_c correspond to the factored dead and live loads.

It is important to note that in each of the three cases above, β_d shall be taken as the following:

$$\beta_d = \frac{\text{Maximum factored sustained axial load}}{\text{Maximum factored axial load}}$$

10.13.7 Moment Magnification for Flexural Members

The strength of a laterally unbraced frame is governed by the stability of the columns and by the degree of end restraint provided by the beams in the frame. If plastic hinges form in the restraining beams, the structure approaches a mechanism and its axial load capacity is drastically reduced. Section 10.13.7 requires that the restraining flexural members (beams or slabs) have the capacity to resist the magnified column moments. The ability of the moment magnifier method to provide a good approximation of the actual magnified moments at the member ends in sway frame is a significant improvement over the reduction factor method for long columns prescribed in earlier ACI codes to account for member slenderness in design.

SUMMARY OF DESIGN EQUATIONS

A summary of the equations for the design of slender columns subjected to dead, live and lateral loads, in both nonsway and sway frames is presented in this section. Examples 11.1 and 11.2 illustrate the application of these equations for the design of columns in nonsway and sway frames, respectively.

- **Nonsway Frames**

1. Determine the factored load combinations per 9.2.

   It is assumed in the examples that follow that the load factor for live load is 0.5 (i.e. condition 9.2.1(a) applies) and that the wind load has been reduced by a directionality factor (9.2.1(b)).

   Note that the factored moments $M_{u,top}$ and $M_{u,bot}$ at the top and bottom of the column, respectively, are to be determined using a first-order frame analysis, based on the cracked section properties of the members.

2. For each load combination, determine M_c, where M_c is the largest factored column end moment, including slenderness effects (if required). Note that M_c may be determined by one of the following methods:

   a. Second-order (P-Δ) analysis (10.10.1)

   b. Magnified moment method (only if $k\ell_u/r \leq 100$; see 10.12 and step (3) below)

   Determine the required column reinforcement for the critical load combination determined in step (1) above. Each load combination consists of P_u and M_c.

3. Magnified moment method (10.12):

   Slenderness effects can be neglected when

$$\frac{k\ell_u}{r} \leq 34 - 12\left(\frac{M_1}{M_2}\right)$$ *Eq. (10-7)*

where $[34-12\,M_1/M_2] \leq 40$. The term M_1/M_2 is positive if the column is bent in single curvature, negative if bent in double curvature. If $M_1 = M_2 = 0$, assume $M_2 = M_{2,\,min}$. In this case $k\ell_u/r = 34.0$.

When slenderness effects need to be considered, determine M_c for each load combination:

$$M_c = \delta_{ns} M_2$$ *Eq. (10-8)*

where

$$M_2 = \text{larger of } M_{u,bot} \text{ and } M_{u,top}$$

$$\geq P_u\,(0.6 + 0.03h)$$ *10.12.3.2*

$$\delta_{ns} = \frac{C_m}{1 - \dfrac{P_u}{0.75P_c}} \geq 1.0$$ *Eq. (10-9)*

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2}$$ *Eq. (10-10)*

$$EI = \frac{(0.2E_cI_g + E_sI_{se})}{1 + \beta_d}$$ *Eq. (10-11)*

or

$$EI = \frac{0.4E_cI_g}{1 + \beta_d}$$ *Eq. (10-12)*

$$\beta_d = \frac{\text{Maximum factored axial sustained load}}{\text{Maximum factored axial load associated with the same load combination}}$$ *10.11.1*

$$C_m = 0.6 + 0.4\left(\frac{M_1}{M_2}\right) \geq 0.4 \quad \text{(for columns without transverse loads)}$$ *Eq. (10-13)*

$$= 1.0 \quad \text{(for columns with transverse loads)}$$

The effective length factor k shall be taken as 1.0, or may be determined from analysis (i.e., alignment chart or equations given in R10.12.1). In the latter case, k shall be based on the E and I values determined according to 10.11.1 (see 10.12.1).

- **Sway Frames**

1. Determine the factored load combinations per 9.2.

   a. Gravity (dead and live) loads

   The moments $(M_{u,bot})_{ns}$ and $(M_{u,top})_{ns}$ at the bottom and top of column, respectively, are to be determined using an elastic first-order frame analysis, based on the cracked section properties of the members.

   The moments M_1 and M_2 are the smaller and the larger of the moments $(M_{u,bot})_{ns}$ and $(M_{u,top})_{ns}$, respectively. The moments M_{1ns} and M_{2ns} are the factored end moments at the ends at which M_1 and M_2 act, respectively.

   b. Gravity (dead and live) plus lateral loads

   The total moments at the top and bottom of the column are $M_{u,top} = (M_{u,top})_{ns} + (M_{u,top})_s$ and $M_{u,bot} = (M_{u,bot})_{ns} + (M_{u,bot})_s$, respectively. The moments M_1 and M_2 are the smaller and the larger of the moments $M_{u,top}$ and $M_{u,bot}$, respectively. Note that at this stage, M_1 and M_2 do not include slenderness effects. The moments M_{1ns} and M_{1s} are the factored nonsway and sway moments, respectively, at the end of the column at which M_1 acts, while M_{2ns} and M_{2s} are the factored nonsway and sway moments, respectively, at the end of the column at which M_2 acts.

   c. Gravity (dead) plus lateral loads

   The definitions for the moments in this load combination are the same as given above for part 1(b).

   d. The effects due to lateral forces acting equal and opposite to the ones in the initial direction of analysis must also be considered in the load combinations given in parts 1(b) and 1(c) above.

2. Determine the required column reinforcement for the critical load combination determined in step (1) above. Each load combination consists of P_u, M_1, and M_2, where now M_1 and M_2 are the total factored end moments, including slenderness effects. Note that if the critical load P_c is computed using EI from Eq. (10-11), it is necessary to estimate first the column reinforcement. Moments M_1 and M_2 are determined by one of the following methods:

   a. Second-order (P-Δ) analysis (10.10.1)

   b. Magnified moment method (only if $k\ell_u/r \le 100$; see 10.13 and step 3 below)

3. Magnified moment method (see 10.13):

   Slenderness effects can be neglected when

$$\frac{k\ell_u}{r} < 22 \qquad \textit{10.13.2}$$

   When slenderness effects need to be considered:

$$M_1 = M_{1ns} + \delta_s M_{1s} \qquad \textit{Eq. (10-15)}$$

$$M_2 = M_{2ns} + \delta_s M_{2s} \qquad \textit{Eq. (10-16)}$$

   The moments $\delta_s M_{1s}$ and $\delta_s M_{2s}$ are to be computed by one of the following methods (10.13.4):

   a. Second-order elastic analysis (see 10.13.4.1)

   b. Approximate second-order analysis (10.13.4.2)

$$\delta_s M_s = \frac{M_s}{1 - Q} \ge M_s, \quad 1.0 \le \delta_s \le 1.5 \qquad \textit{Eq. (10-17)}$$

   where

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us} \ell_c} \qquad \textit{Eq. (10-6)}$$

   c. Approximate magnifier method given in ACI code (see 10.13.4.3):

$$\delta_s M_s = \frac{M_s}{1 - \dfrac{\Sigma P_u}{0.75 \Sigma P_c}} \ge M_s \qquad \textit{Eq. (10-18)}$$

   where

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2} \qquad \textit{Eq. (10-10)}$$

$$EI = \frac{(0.2 E_c I_g + E_s I_{se})}{1 + \beta_d} \qquad \textit{Eq. (10-11)}$$

   or

$$EI = \frac{0.4E_cI_g}{1 + \beta_d}$$ *Eq. (10-12)*

The effective length factor k must be greater than 1.0 and shall be based on the E and I values determined according to 10.11.1 (see 10.13.1).

4. Check if the maximum moment occurs at the ends of the column or between the ends of the column (10.13.5). If

$$\frac{\ell_u}{r} > \frac{35}{\sqrt{\frac{P_u}{f'_c A_g}}}$$ *Eq. (10-19)*

the column must be designed for the factored axial load P_u and the moment M_c, where

$$M_c = \delta_{ns} M_2 = \delta_{ns}(M_{2ns} + \delta_s M_{2s})$$

where

$$\delta_{ns} = \frac{C_m}{1 - \frac{P_u}{0.75P_c}}$$

To calculate δ_{ns} k is determined according to the provisions in 10.12.1 and $\delta_{ns} \geq 1.0$ and M_1 and M_2 given by Eqs. (10-15), (10-16) are used to calculate C_m according to Eq. (10-13).

5. Check the possibility of sidesway instability under gravity loads (10.13.6):

a. When $\delta_s M_s$ is computed from 10.13.4.1:

$$\frac{\text{Second-order lateral deflections}}{\text{First-order lateral deflections}} \leq 2.5$$

based on factored dead and live loads plus factored lateral oad.

b. When $\delta_s M_s$ is computed from 10.13.4.2:

$$Q = \frac{\Sigma P_u \Delta_o}{V_u \ell_c} \leq 0.60$$

based on factored dead and live loads.

c. When $\delta_s M_s$ is computed from 10.13.4.3:

$$0 < \delta_s \leq 2.5$$

where δ_s is computed using ΣP_u and ΣP_c corresponding to the factored dead and live loads.

In all three cases, β_d shall be taken as:

$$\beta_d = \frac{\text{Maximum factored sustained axial load}}{\text{Maximum factored axial load}}$$

Reference 11.1 gives the derivation of the design equations for the slenderness provisions outlined above.

REFERENCES

11.1 MacGregor, J. G., "Design of Slender Concrete Columns—Revisited," *ACI Structural Journal,* V. 90, No. 3, May-June 1993, pp. 302-309.

11.2 pcaColumn—Design and Investigation of Reinforced Concrete Column Sections, Portland Cement Association, Skokie, IL 2005.

Example 11.1—Slenderness Effects for Columns in a Nonsway Frame

Design columns A3 and C3 in the first story of the 10-story office building shown below. The clear height of the first story is 21 ft-4 in., and is 11 ft-4 in. for all of the other stories. Assume that the lateral load effects on the building are caused by wind, and that the dead loads are the only sustained loads. Other pertinent design data for the building are as follows:

Material properties:

Concrete:

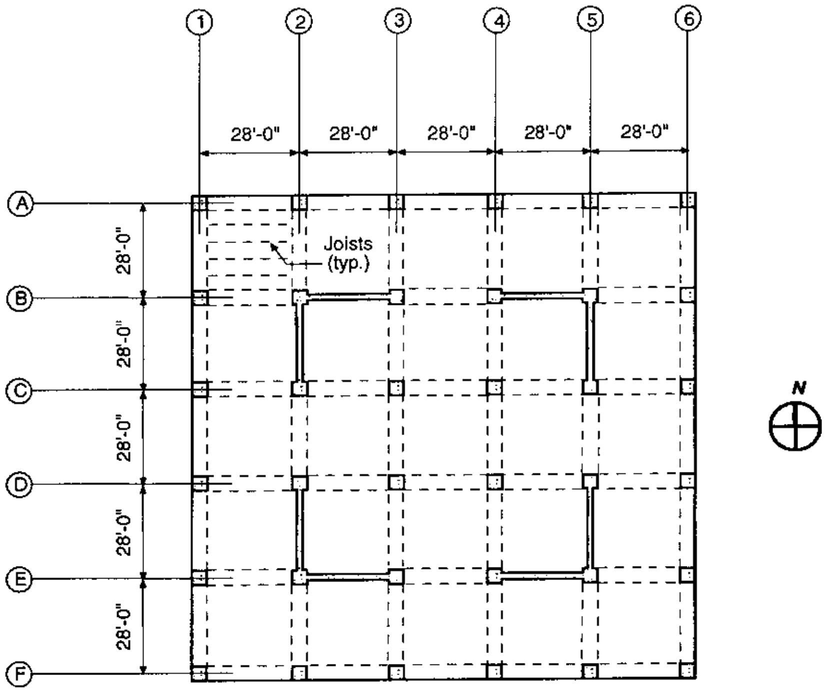
Floors: f'_c = 4,000 psi, w_c = 150 pcf

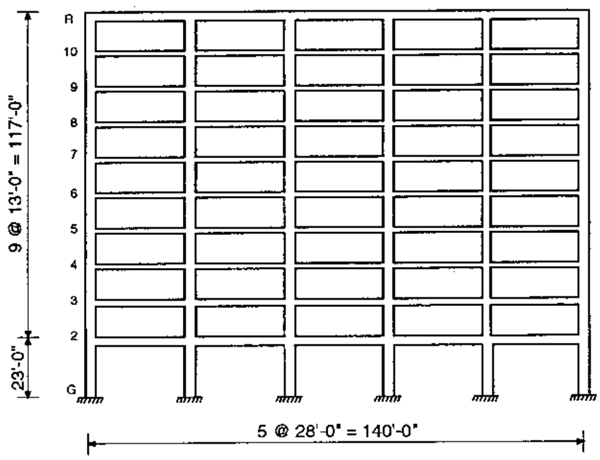
Columns and walls: f'_c = 6,000 psi, w_c = 150 pcf

Reinforcement: f_y = 60 ksi

Beams: 24 × 20 in.
Exterior columns: 20 × 20 in.
Interior columns: 24 × 24 in.
Shearwalls: 12 in.

Weight of floor joists = 86 psf
Superimposed dead load = 32 psf
Roof live load = 30 psf
Floor live load = 50 psf
Wind loads computed according to ASCE 7.





Example 11.1 (cont'd)	Calculations and Discussion	Code Reference

1. Factored axial loads and bending moments for columns A3 and C3 in the first story

Column A3

Eq.	No.	Load Case	Axial Load (kips)	Bending Moment (ft-kips) Top	Bottom
		Dead (D)	718.0	79.0	40.0
		Live (L)*	80.0	30.3	15.3
		Roof live load (L_r)	12.0	0.0	0.0
		Wind (W)	±8.0	±1.1	±4.3
Eq.	No.	Load Combination			
9-1	1	1.4D	1,005.2	110.6	56.0
9-2	2	1.2D + 1.6L + 0.5L_r	995.6	143.3	72.5
9-3	3	1.2D + 0.5L + 1.6L_r	920.8	110.0	55.7
	4	1.2D + 1.6L_r + 0.8W	887.2	95.7	51.4
	5	1.2D + 1.6L_r - 0.8W	874.4	93.9	44.6
9-4	6	1.2D + 0.5L + 0.5L_r + 1.6W	920.4	111.7	62.5
	7	1.2D + 0.5L + 0.5L_r - 1.6W	894.8	108.2	48.8
9-6	8	0.9D + 1.6W	659.0	72.9	42.9
	9	0.9D - 1.6W	633.4	69.3	29.1

*includes live load reduction per ASCE 7

Column C3

Eq.	No.	Load Case	Axial Load (kips)	Bending Moment (ft-kips) Top	Bottom
		Dead (D)	1,269.0	1.0	0.7
		Live (L)*	147.0	32.4	16.3
		Roof live load (L_r)	24.0	0.0	0.0
		Wind (W)	±3.0	±2.5	±7.7
Eq.	No.	Load Combination			
9-1	1	1.4D	1,776.6	1.4	1.0
9-2	2	1.2D + 1.6L + 0.5L_r	1,770.0	53.0	26.9
9-3	3	1.2D + 0.5L + 1.6L_r	1,634.7	17.4	9.0
	4	1.2D + 1.6L_r + 0.8W	1,563.6	3.2	7.0
	5	1.2D + 1.6L_r - 0.8W	1,558.8	-0.8	-5.3
9-4	6	1.2D + 0.5L + 0.5L_r + 1.6W	1,613.1	21.4	21.3
	7	1.2D + 0.5L + 0.5L_r - 1.6W	1,603.5	13.4	-3.3
9-6	8	0.9D + 1.6W	1,146.9	4.9	13.0
	9	0.9D - 1.6W	1,137.3	-3.1	-11.7

*includes live load reduction per ASCE 7

Note that Columns A3 and C3 are bent in double curvature with the exception of Load Case 7 for Column C3.

2. Determine if the frame at the first story is nonsway or sway

The results from an elastic first-order analysis using the section properties prescribed in 10.11.1 are as follows:

ΣP_u = total vertical load in the first story corresponding to the lateral loading case for which ΣP_u is greatest

The total building loads are: D = 37,371 kips, L = 3609 kips, and L_r = 605 kips.
The maximum $\sum P_u$ is determined from Eq. (9-4):

$$\Sigma P_u = (1.2 \times 37{,}371) + (0.5 \times 3609) + (0.5 \times 605) + 0 = 46{,}952 \text{ kips}$$

V_{us} = factored story shear in the first story corresponding to the wind loads
$= 1.6 \times 324.3 = 518.9$ kips — *Eq. (9-4), (9-6)*

Δ_o = first-order relative lateral deflection between the top and bottom of the first story due to V_{us}
$= 1.6 \times (0.03\text{-}0) = 0.05$ in.

$$\text{Stability index } Q = \frac{\sum P_u \Delta_o}{V_{us}\ell_c} = \frac{46{,}952 \times 0.05}{518.9 \times [(23 \times 12)-(20/2)]} = 0.02 < 0.05$$ *Eq. (10-6)*

Since Q < 0.05, the frame at the first story level is considered nonsway. *10.11.4.2*

3. Design of column C3

Determine if slenderness effects must be considered.

Using an effective length factor k = 1.0, *10.12.1*

$$\frac{k\ell_u}{r} = \frac{1.0 \times 21.33 \times 12}{0.3 \times 24} = 35.6$$

The following table contains the slenderness limit for each load case:

Eq.	No.	Axial loads (kips)	Bending Moment (ft-kips)		Curvature	M_1 (ft-kips)	M_2 (ft-kips)	M_1/M_2	Slenderness limit
		P_u	M_{top}	M_{bot}					
9-1	1	1776.6	1.4	1.0	Double	1.0	1.4	0.70	40.00
9-2	2	1770.0	53.0	26.9	Double	26.9	53.0	0.51	40.00
9-3	3	1634.7	17.4	9.0	Double	9.0	17.4	0.52	40.00
	4	1564.2	3.7	8.5	Double	3.7	8.5	0.43	39.20
	5	1558.2	-1.3	-6.9	Double	1.3	6.9	0.19	36.27
9-4	6	1613.1	21.4	21.3	Double	21.3	21.4	1.00	40.00
	7	1603.5	13.4	-3.3	Single	3.3	13.4	-0.25	31.02
9-6	8	1146.9	4.9	13.0	Double	4.9	13.0	0.38	38.54
	9	1137.3	-3.1	-11.7	Double	3.1	11.7	0.27	37.18

The least value of $34-12\left(\frac{M_1}{M_2}\right)$ is
obtained from load combination no. 7:

$$34 - 12\left(\frac{M_1}{M_2}\right) = 34 - 12\left(\frac{3.3}{13.4}\right) = 31.02 < 40$$

Slenderness effects need to be considered for column C3 since $k\ell_u/r > 34 - 12\,(M_1/M_2)$. 10.12.2

The following calculations illustrate the magnified moment calculations for load combination no. 7:

$$M_c = \delta_{ns} M_2$$

where

$$\delta_{ns} = \frac{C_m}{1 - \dfrac{P_u}{0.75P_c}} \geq 1$$

$$C_m = 0.6 + 0.4\left(\frac{M_1}{M_2}\right) \geq 0.40$$

$$= 0.6 + 0.4\left(\frac{3.3}{13.4}\right) = 0.70$$

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2}$$

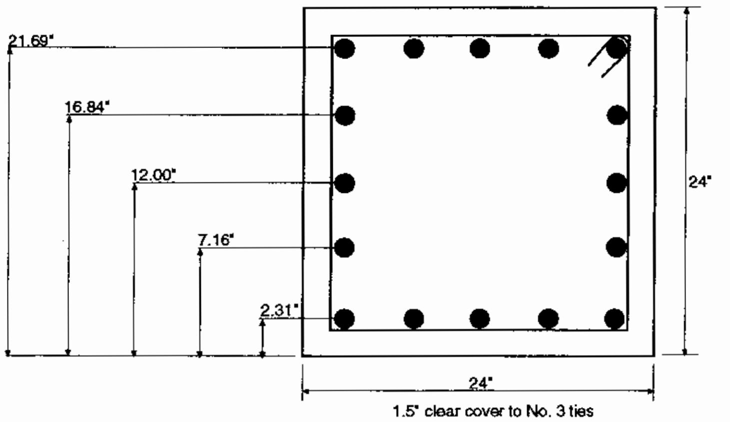
$$EI = \frac{0.2E_cI_g + E_sI_e}{1+\beta_d}$$

$$E_c = 57{,}000\,\frac{\sqrt{6000}}{1000} = 4415 \text{ ksi}$$

$$I_g = \frac{24^4}{12} = 27{,}648 \text{ in.}^4$$

$$E_s = 29{,}000 \text{ ksi}$$

Assuming 16-No. 7 bars with 1.5 cover to No. 3 ties as shown in the figure.



$$I_{se} = 2\left[(5 \times 0.6)(21.69-12)^2 + (2 \times 0.6)(16.84-12)^2\right]$$

$$= 619.6 \text{ in.}^4$$

Since the dead load is the only sustained load,

$$\beta_d = \frac{1.2P_D}{1.2P_D + 0.5P_L + 0.5P_{Lr} - 1.6W}$$

$$= \frac{1.2 \times 1269}{(1.2 \times 1269) + (0.5 \times 147) + (0.5 \times 24) - (1.6 \times 3)}$$

$$= 0.95$$

$$EI = \frac{(0.2 \times 4415 \times 27{,}648) + (29{,}000 \times 619.6)}{1 + 0.95} = 21.73 \times 10^6 \text{ kip-in.}^2$$

$$P_c = \frac{\pi^2 \times 21.73 \times 10^6}{(1 \times 21.33 \times 12)^2} = 3274 \text{ kips}$$

$$\delta_{ns} = \frac{0.7}{1 - \dfrac{1603.5}{0.75 \times 3274}} = 2.02$$

Check miminum moment requirement:

$$M_{2,\ min} = P_n(0.6 + 0.03h)$$

$$= 1603.5[0.6 + (0.03 \times 24)]/12$$

$$= 176.4 \text{ ft-kip} > M_2$$

$$M_c = 2.02 \times 176.4 = 356.3 \text{ ft-kip}$$

The following table contains results from a strain compatibility analysis, where compressive strains are taken as positive (see Part 6 and 7).

Therefore, since $fM_n > M_u$ for all $fP_n = P_u$, use a 24 3 24 in. column with 16-No. 7 bars (r_g = 1.7%).

No.	P_u (kips)	M_u (ft-kips)	c (in)	ε_t	ϕ	ϕP_n (kips)	ϕM_n (ft-kips)
1	1776.6	1.4	25.92	0.00049	0.65	1776.6	367.2
2	1770.0	53.0	25.83	0.00048	0.65	1770.0	371.0
3	1634.7	17.4	23.86	0.00027	0.65	1634.7	447.0
4	1563.6	7.0	22.85	0.00015	0.65	1563.6	480.9
5	1558.8	5.3	22.78	0.00014	0.65	1558.8	483.2
6	1613.1	21.4	23.55	0.00024	0.65	1613.1	457.8
7	1603.5	356.3	23.41	0.00022	0.65	1603.5	462.5
8	1146.9	13.0	17.25	-0.00077	0.65	1146.9	609.9
9	1137.3	11.7	17.13	-0.00080	0.65	1137.3	611.7

Design for P_u = and M_c = can be performed manually, by creating an interaction diagram as shown in example 6.4. For this example, Figure 11-14 shows the design srength interaction diagram for Column C3 obtained from the computer program pcaColumn. The figure also shows the axial load and moments for all load combinations.

4. Design of column A3

a. Determine if slenderness effects must be considered.

Determine k from the alignment chart of Fig. 11-10 or from Fig. R10.12.1:

$$I_{col} = 0.7\left(\frac{20^4}{12}\right) = 9{,}333 \text{ in.}^4$$ *10.11.1(b)*

$$E_c = 57{,}000\ \frac{\sqrt{6{,}000}}{1{,}000} = 4{,}415 \text{ ksi}$$ *8.5.1*

For the column below level 2:

$$\left(\frac{E_cI}{\ell_c}\right) = \frac{4{,}415\times 9{,}333}{[(23\times 12)-(20/2)]} = 155 \times 10^3 \text{ in.-kips}$$

For the column above level 2:

$$\left(\frac{E_cI}{\ell_c}\right) = \frac{4{,}415\times 9{,}333}{13\times 12} = 264 \times 10^3 \text{ in.-kips}$$

$$I_{beam} = 0.35\left(\frac{24\times 20^3}{12}\right) = 5{,}600 \text{ in.}^4$$ *10.11.1(b)*

$$\frac{EI}{\ell} = \frac{57\sqrt{4{,}000}\times 5{,}600}{28\times 12} = 60 \times 10^3 \text{ in.-kips}$$

$$\psi_A = \frac{\Sigma E_cI/\ell_c}{\Sigma E_cI/\ell} = \frac{155+264}{60} = 7.0$$

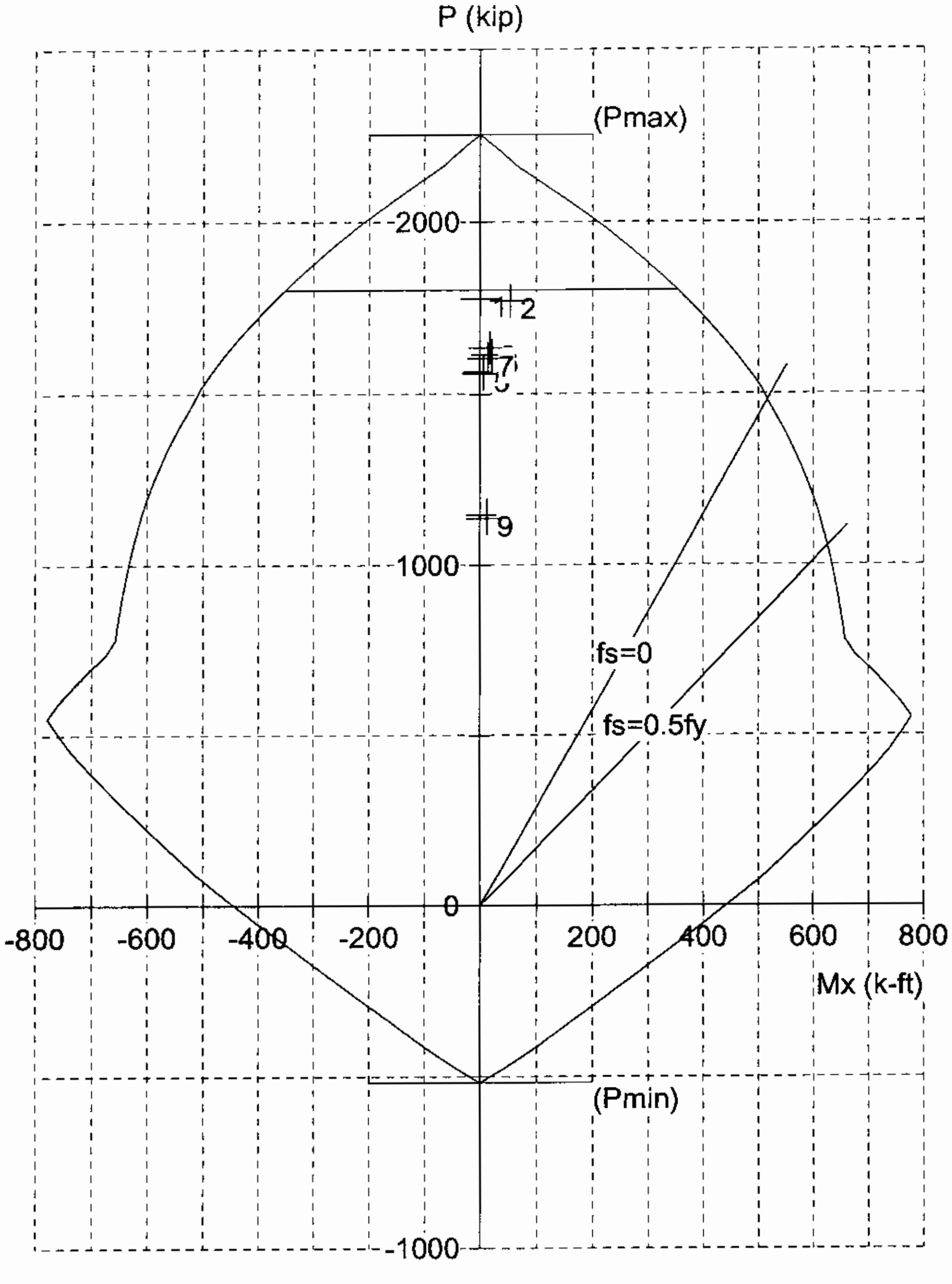


Figure 11-14 Interaction Diagram for Column C3

Assume ψ_B = 1.0 (column essentially fixed at base)

From Fig. R10.12.1(a), k = 0.86.

Therefore, for column A3 bent in double curvature, the least $34-12\left(\frac{M_1}{M_2}\right)$ is obtained from load combination no. 9:

$$34-12\left(\frac{-29.1}{69.3}\right) = 39.0$$

$$\frac{k\ell_u}{r} = \frac{0.86\times 21.33\times 12}{0.3\times 20} = 36.7 < 39.0$$

For column A3 bent in single curvature, the least $34-12\left(\frac{M_1}{M_2}\right)$ is obtained from load combination no. 8:

$$\frac{k\ell_u}{r} = 36.7 > 34 - 12\left(\frac{42.9}{72.9}\right) = 26.9$$

Therefore, column slenderness need not be considered for column A3 if bent in double curvature. However, to illustrate the design procedure including slenderness effects for nonsway columns, assume single curvature bending.

b. Determine total moment M_c (including slenderness effects) for each load combination.

$$M_c = \delta_{ns} M_2$$ *Eq. (10-8)*

where

$$\delta_{ns} = \frac{C_m}{1 - \frac{P_u}{0.75P_c}} \geq 1.0$$ *Eq. (10-9)*

The following table summarizes magnified moment computations for column A3 for all load combinations, followed by detailed calculations for combination no. 6 to illustrate the procedure.

No.	Pu (kips)	M_2 (ft-kips)	b	EI x 10^6 (kip-in^2)	P_c kips	C_m	d_{ns}	$M_{2,min}$ (ft-kips)	M_c (ft-kips)
1	1005.2	110.6	1.00	9.88	2013	0.80	2.40	100.5	265.6
2	995.6	143.3	0.87	10.60	2158	0.80	2.08	99.6	298.6
3	920.8	110.0	0.94	10.21	2080	0.80	1.96	92.1	215.3
4	887.2	95.7	0.97	10.03	2042	0.82	1.94	88.7	185.3
5	874.4	93.9	0.99	9.96	2028	0.79	1.86	87.4	174.5
6	920.4	111.7	0.94	10.21	2079	0.82	2.01	92.0	224.6
7	894.8	108.2	0.96	10.07	2051	0.78	1.87	89.5	201.8
8	659.0	72.9	0.98	9.98	2033	0.84	1.47	65.9	107.2
9	633.4	69.3	1.00	9.89	2014	0.77	1.32	63.3	91.7

Load combination no. 6:

$$U = 1.2D + 0.5L + 0.5\,L_r + 1.6W$$

$$C_m = 0.6 + 0.4\left(\frac{M_1}{M_2}\right) \geq 0.4$$ *Eq. (10-13)*

$$= 0.6 + 0.4\left(\frac{62.5}{111.7}\right) = 0.82$$

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2}$$ *Eq. (10-10)*

$$EI = \frac{(0.2E_cI_g + E_sI_{se})}{1 + \beta_d}$$ *Eq. (10-11)*

$$E_c = 57{,}000\,\frac{\sqrt{6{,}000}}{1{,}000} = 4{,}415 \text{ ksi}$$ *8.5.1*

$$I_g = \frac{20^4}{12} = 13{,}333 \text{ in.}^4$$

$$E_s = 29{,}000 \text{ ksi}$$ *8.5.2*

Assuming 8-No. 8 bars with 1.5 in. cover to No. 3 ties:

$$I_{se} = 2\left[(3\times 0.79)\left(\frac{20}{2} - 1.5 - 0.375 - \frac{1.00}{2}\right)^2\right] = 276 \text{ in.}^4$$

Since the dead load is the only sustained load,

$$\beta_d = \frac{1.2P_D}{1.2P_D + 0.5P_L + 0.5P_{L_r} + 1.6P_w}$$

$$= \frac{1.2\times 718}{(1.2\times 718) + (0.5\times 80) + (0.5\times 12) + (1.6\times 8)} = 0.94$$

$$EI = \frac{(0.2\times 4{,}415\times 13{,}333) + (29{,}000\times 276)}{1 + 0.94} = 10.21\times 10^6 \text{ kip-in.}^2$$

From Eq. (10-12):

$$EI = \frac{0.4E_cI_g}{1 + \beta_d}$$

$$= \frac{0.4 \times 4{,}415 \times 13{,}333}{1+0.94} = 12.14 \times 10^6 \text{ kip-in.}^2$$

Using EI from Eq. (10-10), the critical load P_c is:

$$P_c = \frac{\pi^2 \times 10.21 \times 10^6}{(0.86 \times 21.33 \times 12)^2} = 2{,}079 \text{ kips}$$

Therefore, the moment magnification factor δ_{ns} is:

$$\delta_{ns} = \frac{0.82}{1 - \dfrac{920.4}{0.75 \times 2{,}079}} = 2.01$$

Check minimum moment requirement:

$M_{2,min} = P_u (0.6 + 0.03h)$ *Eq. (10-14)*

$= 920.4 [0.6 + (0.03 \times 20)]/12$

$= 92.0$ ft-kips $< M_2 = 111.7$ ft-kips

Therefore,

$M_c = 2.01 \times 111.7 = 224.6$ ft-kips

c. Determine required reinforcement.

For the 20 x 20 in. column, try 8-No. 8 bars.

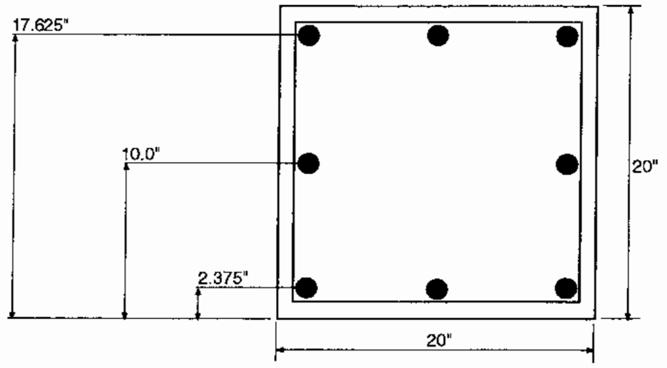
Determine maximum allowable axial compressive force, $\phi P_{n,max}$:

$\phi P_{n,max} = 0.80\phi \left[0.85f'_c(A_g - A_{st}) + f_yA_{st}\right]$ *Eq. (10-2)*

$= (0.80 \times 0.65)\left[(0.85 \times 6)(20^2 - 6.32) + (60 \times 6.32)\right]$

$= 1{,}241.2$ kips $>$ maximum $P_u = 1{,}005.2$ kips O.K.

The following table contains results from a strain compatibility analysis, where compressive strains are taken as positive (see Parts 6 and 7).



No.	P_u (kips)	M_u (ft-kips)	c (in.)	ε_t	ϕ	ϕP_n (kips)	ϕM_n (ft-kips)
1	1,005.2	265.6	17.81	0.00003	0.65	1,005.2	298.0
2	995.6	298.6	17.64	0.00000	0.65	995.6	301.1
3	920.8	215.3	16.42	-0.00022	0.65	920.8	321.4
4	887.2	185.3	15.88	-0.00033	0.65	887.2	329.3
5	874.4	174.5	15.67	-0.00037	0.65	874.4	332.1
6	920.4	224.6	16.41	-0.00022	0.65	920.4	321.6
7	894.8	201.8	16.00	-0.00030	0.65	894.8	327.6
8	659.0	107.2	12.36	-0.00128	0.65	659.0	364.8
9	633.4	92.4	12.00	-0.00141	0.65	633.4	367.2

Therefore, since $\phi M_n > M_u$ for all $\phi P_n = P_u$, use a 20 × 20 in. column with 8-No. 8 bars (ρ_g = 1.6%). Figure 11-15 obtained from pcaColumn [11.2], contains the design strength interaction diagram for Column A3 with the factored axial loads and magnified moments for all load combinations.

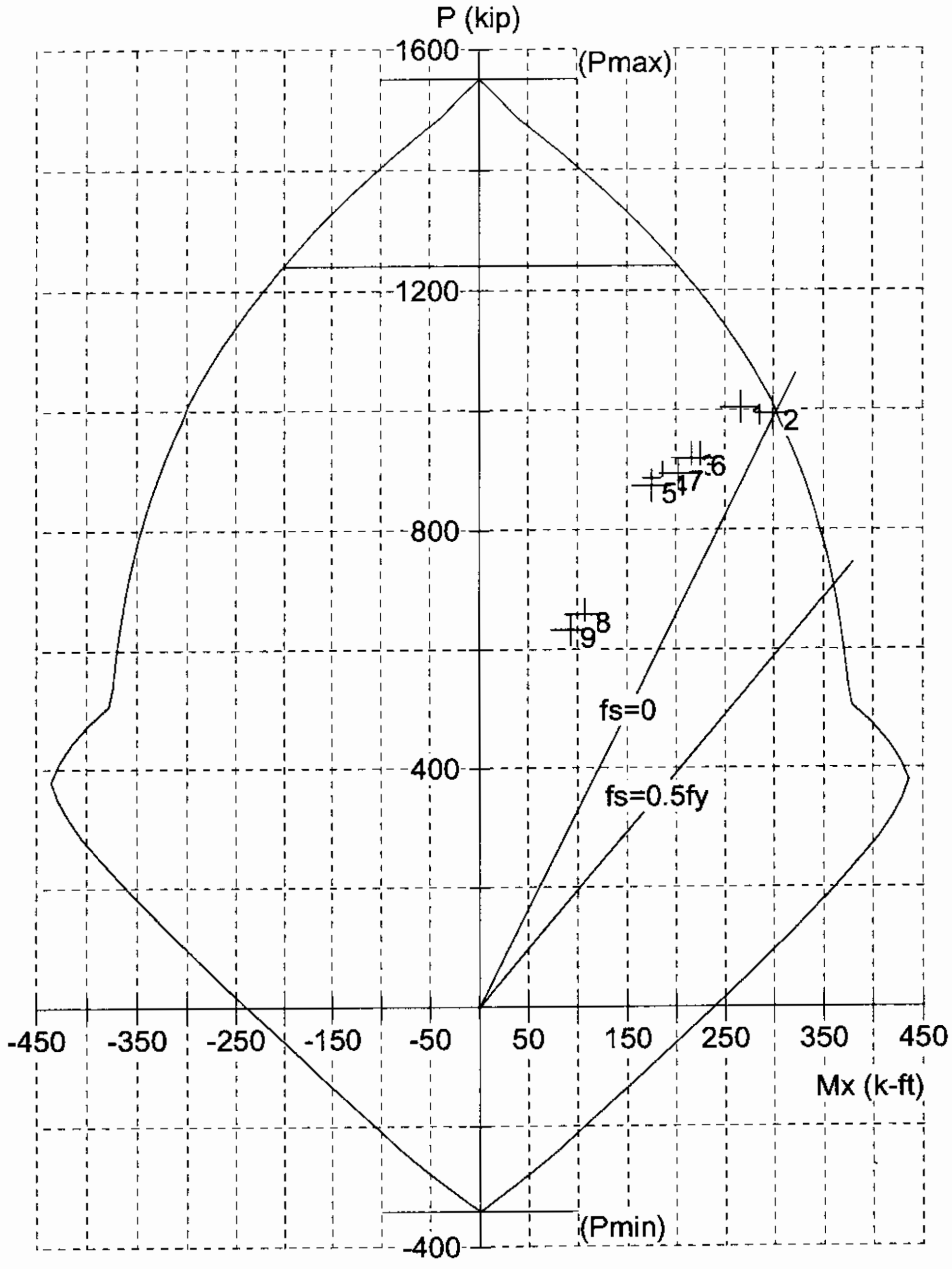


Figure 11-15 Design Strength Interaction Diagram for Column A3

Example 11.2—Slenderness Effects for Columns in a Sway Frame

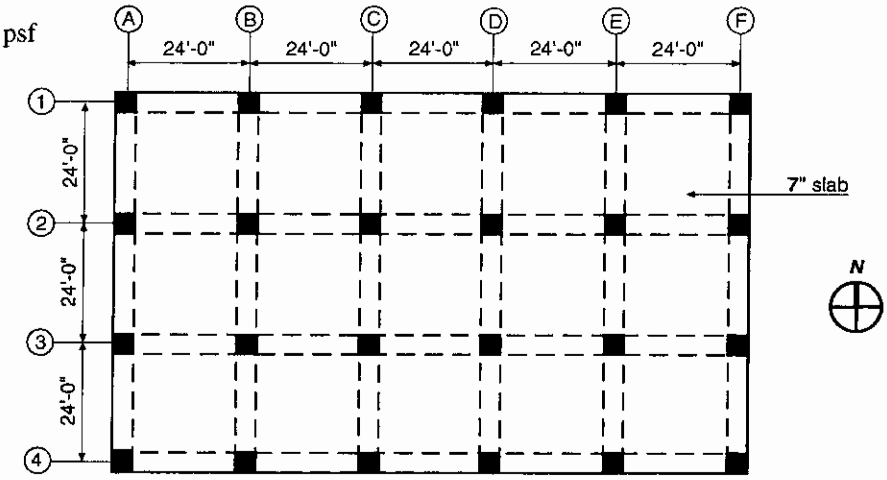
Design columns C1 and C2 in the first story of the 12-story office building shown below. The clear height of the first story is 13 ft-4 in., and is 10 ft-4 in. for all of the other stories. Assume that the lateral load effects on the building are caused by wind, and that the dead loads are the only sustained loads. Other pertinent design data for the building are as follows:

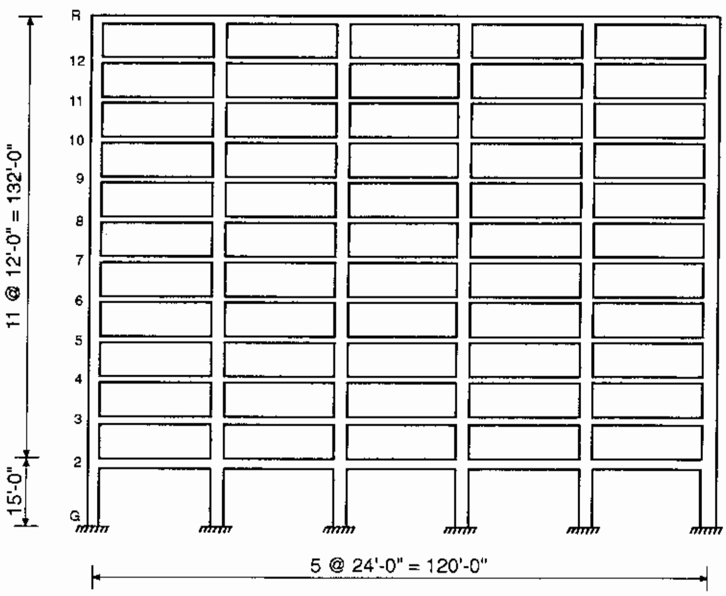
Material properties:

Concrete: = 6,000 psi for columns in the bottom two stories (w_c = 150 pcf)
= 4,000 psi elsewhere (w_c = 150 pcf)
Reinforcement: f_y = 60 ksi

Beams: 24 × 20 in.
Exterior columns: 22 × 22 in.
Interior columns: 24 × 24 in.

Superimposed dead load = 30 psf
Roof live load = 30 psf
Floor live load = 50 psf
Wind loads computed according to ASCE 7





1. Factored axial loads and bending moments for columns C1 and C2 in the first story

Since this is a symmetrical frame, the gravity loads will not cause appreciable sidesway.

Column C1

	No.	Load Case	Axial Load (kips)	Bending moment (ft-kips) Top	Bottom						
		Dead (D)	622.4	34.8	17.6						
		Live (L)*	73.9	15.4	7.7						
		Roof live load (L_r)	8.6	0.0	0.0						
		Wind (W) (N-S)	-48.3	17.1	138.0						
		Wind (W) (S-N)	48.3	-17.1	-138.0						
	No.	Load Combination				M_1	M_2	M_{1ns}	M_{2ns}	M_{1s}	M_{2s}
9-1	1	1.4D	871.4	48.7	24.6	24.6	48.7	24.6	48.7	---	---
9-2	2	1.2D + 1.6L + 0.5L_r	869.4	66.4	33.4	33.4	66.4	33.4	66.4	---	---
9-3	3	1.2D + 0.5L + 1.6L_r	797.6	49.5	25.0	25.0	49.5	25.0	49.5	---	---
	4	1.2D + 1.6L_r + 0.8W	722.0	55.4	131.5	55.4	131.5	41.8	21.1	13.7	110.4
	5	1.2D + 1.6L_r - 0.8W	799.3	28.1	-89.3	28.1	-89.3	41.8	21.1	-13.7	-110.4
9-4	6	1.2D + 0.5L + 0.5L_r + 1.6W	710.9	76.8	245.8	76.8	245.8	49.5	25.0	27.4	220.8
	7	1.2D + 0.5L + 0.5L_r - 1.6W	865.4	22.1	-195.8	22.1	-195.8	49.5	25.0	-27.4	-220.8
9-6	8	0.9D + 1.6W	482.9	58.7	236.6	58.7	236.6	31.3	15.8	27.4	220.8
	9	0.9D - 1.6W	637.4	4.0	-205.0	4.0	-205.0	31.3	15.8	-27.4	-220.8

*includes live load reduction per ASCE 7

Column C2

	No.	Load Case	Axial Load (kips)	Bending moment (ft-kips) Top	Bottom						
		Dead (D)	1,087.6	-2.0	-1.0						
		Live (L)*	134.5	-15.6	-7.8						
		Roof live load (L_r)	17.3	0.0	0.0						
		Wind (W) (N-S)	-0.3	43.5	205.0						
		Wind (W) (S-N)	0.3	-43.5	-205.0						
	No.	Load Combination				M_1	M_2	M_{1ns}	M_{2ns}	M_{1s}	M_{2s}
9-1	1	1.4D	1,522.6	-2.8	-1.4	-1.4	-2.8	-1.4	-2.8	---	---
9-2	2	1.2D + 1.6L + 0.5L_r	1,529.0	-27.4	-13.7	-13.7	-27.4	-13.7	-27.4	---	---
9-3	3	1.2D + 0.5L + 1.6L_r	1,400.1	-10.2	-5.1	-5.1	-10.2	-5.1	-10.2	---	0.0
	4	1.2D + 1.6L_r + 0.8W	1,332.6	32.4	162.8	32.4	162.8	-2.4	-1.2	34.8	164.0
	5	1.2D + 1.6L_r - 0.8W	1,333.0	-37.2	-165.2	-37.2	-165.2	-2.4	-1.2	-34.8	-164.0
9-4	6	1.2D + 0.5L + 0.5L_r + 1.6W	1,380.5	59.4	322.9	59.4	322.9	-10.2	-5.1	69.6	328.0
	7	1.2D + 0.5L + 0.5L_r - 1.6W	1,381.5	-79.8	-333.1	-79.8	-333.1	-10.2	-5.1	-69.6	-328.0
9-6	8	0.9D + 1.6W	978.4	67.8	327.1	67.8	327.1	-1.8	-0.9	69.6	328.0
	9	0.9D - 1.6W	979.3	-71.4	-328.9	-71.4	-328.9	-1.8	-0.9	-69.6	-328.0

*includes live load reduction per ASCE 7

2. Determine if the frame at the first story is nonsway or sway

The results from an elastic first-order analysis using the section properties prescribed in 10.11.1 are as follows:

ΣP_u = total vertical load in the first story corresponding to the lateral loading case for which ΣP_u is greatest

The total building loads are: D = 17,895 kips, L = 1,991 kips, L_T = 270 kips. The maximum ΣP_u is from Eq. (9-4):

$\Sigma P_u = (1.2 \times 17{,}895) + (0.5 \times 1{,}991) + (0.5 \times 270) + 0 = 22{,}605$ kips

V_{us} = factored story shear in the first story corresponding to the wind loads

= 1.6 × 302.6 = 484.2 kips *Eq. (9-4), (9-6)*

Δ_o = first-order relative deflection between the top and bottom of the first story due to V_u
= 1.6 × (0.28 – 0) = 0.45 in.

$$\text{Stability index } Q = \frac{\Sigma P_u \Delta_o}{V_{us}\ell_c} = \frac{22{,}605 \times 0.45}{484.2 \times [(15 \times 12) - (20/2)]} = 0.12 > 0.05$$ *Eq. (10-6)*

Since Q > 0.05, the frame at the first story level is considered sway. *10.11.4.2*

3. Design of column C1

   a. Determine if slenderness effects must be considered.

   Determine k from alignment chart in R10.12.1.

$$I_{col} = 0.7\left(\frac{22^4}{12}\right) = 13{,}665 \text{ in.}^4$$ *10.11.1*

$$E_c = 57{,}000\,\frac{\sqrt{6{,}000}}{1{,}000} = 4{,}415 \text{ ksi}$$ *8.5.1*

For the column below level 2:

$$\frac{E_cI}{\ell_c} = \frac{4{,}415 \times 13{,}665}{(15 \times 12) - 10} = 355 \times 10^3 \text{ in.-kips}$$

For the column above level 2:

$$\frac{E_cI}{\ell_c} = \frac{4{,}415 \times 13{,}665}{12 \times 12} = 419 \times 10^3 \text{ in.-kips}$$

$$I_{beam} = 0.35\left(\frac{24 \times 20^3}{12}\right) = 5{,}600 \text{ in.}^4$$ *10.11.1*

$$\text{For the beam: } \frac{E_cI}{\ell_c} = \frac{57\sqrt{4{,}000} \times 5{,}600}{24 \times 12} = 70 \times 10^3 \text{ in.-kips}$$

$$\psi_A = \frac{\Sigma E_cI/\ell_c}{\Sigma E_cI/\ell} = \frac{355 + 419}{70} = 11.1$$

Assume ψ_B = 1.0 (column essentially fixed at base)

From the alignment chart (Fig. R10.12.1(b)), k = 1.9.

$$\frac{k\ell_u}{r} = \frac{1.9 \times 13.33 \times 12}{0.3 \times 22} = 46 > 22$$ *10.13.2*

Thus, slenderness effects must be considered.

b. Determine total moment M_2 (including slenderness effects) and the design load combinations, using the approximate analysis of 10.13.4.2.

The following table summarizes magnified moment computations for column C1 for all load combinations, followed by detailed calculations for combinations no. 4 and 5 to illustrate the procedure.

No.	Load Combination	ΣP_u (kips)	Δ_o (in)	V_{us} (kips)	Q	δ_s	M_{2ns} (ft-kips)	M_{2s} (ft-kips)	M_2 (ft-kips)
1	1.4D	25,053					48.7		48.7
2	1.2D+1.6L+0.5Lr	24,795					66.4		66.4
3	1.2D+0.5L+1.6Lr	22,903					49.5		49.5
4	1.2D+1.6Lr+0.8W	21,908	0.28	302.6	0.12	1.14	21.1	110.4	147.0
5	1.2D+1.6Lr-0.8W	21,908	0.28	302.6	0.12	1.14	21.1	-110.4	-104.8
6	1.2D+0.5L+0.5Lr+1.6W	22,605	0.28	484.2	0.08	1.08	25.0	220.8	264.2
7	1.2D+0.5L+0.5Lr-1.6W	22,605	0.45	484.2	0.12	1.14	25.0	-220.8	-226.8
8	0.9D+1.6W	16,106	0.45	484.2	0.09	1.10	15.8	220.8	257.9
9	0.9D-1.6W	16,106	0.45	484.2	0.09	1.10	15.8	-220.8	-226.2

$$M_2 = M_{2ns} + M_{2s}$$ *Eq. (10-16)*

$$\delta_s M_{2s} = \frac{M_{2s}}{1-Q} \geq M_{2s}$$ *Eq. (10-17)*

For load combinations no. 4 and 5:

$$U = 1.2D + 1.6L_r \pm 0.8W$$

$$\Sigma P_u = (1.2 \times 17{,}895) + (1.6 \times 270) \pm 0 = 21{,}906 \text{ kips}$$

$$\Delta_o = 0.8 \times (0.28 - 0) = 0.22 \text{ in.}$$

$$V_{us} = 0.8 \times 302.6 = 240.1 \text{ kips}$$

$$\ell_c = (15 \times 12) - (20/2) = 170 \text{ in.}$$

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us}\ell_c} = \frac{21{,}906 \times 0.22}{240.1 \times 170} = 0.12$$

$$\delta = \frac{1}{1-Q} = \frac{1}{1-0.12} = 1.14$$

- For sidesway from north to south (load combination no. 4):

  $\delta_s M_{2s} = 1.14 \times 110.4 = 125.9$ ft-kips

  $M_2 = M_{2ns} + \delta_s M_{2s} = 21.1 + 125.9 = 147.0$ ft-kips

  $P_u = 722.0$ kips

- For sidesway from south to north (load combination no. 5):

  $M_{2s} = 0.8 \times 138.0 = 110.4$ ft-kips

  $M_{2su} = 1.2 \times 17.6 + 1.6 \times 0 = 21.1$ ft-kips

  $\delta_s M_{2s} = 1.14 \times (-110.4) = -125.9$ ft-kips

  $M_2 = 21.1 - 125.9 = -104.8$ ft-kips

  $P_u = 799.3$ kips

c. For comparison purposes, recompute $\delta_s M_{2s}$ using the magnified moment method outlined in 10.13.4.3

$$\delta_s M_{2s} = \frac{M_{2s}}{1 - \dfrac{\Sigma P_u}{0.75 \Sigma P_c}} \geq M_{2s}$$ *Eq. (10-18)*

The critical load P_c is calculated from Eq. (10-10) using k from 10.13.1 and EI from Eq. (10-11) or (10-12). Since the reinforcement is not known as of yet, use Eq. (10-12) to determine EI.

For each of the 12 exterior columns along column lines 1 and 4 (i.e., the columns with one beam framing into them in the direction of analysis), k was determined in part 3(a) above to be 1.9.

$$EI = \frac{0.4 E_c I}{1 + \beta_d} = \frac{0.4 \times 4{,}415 \times 22^4}{12(1 + 0)} = 34.5 \times 10^6 \text{ in.}^2 - \text{kips}$$ *Eq. (10-12)*

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2} = \frac{\pi^2 \times 34.5 \times 10^6}{(1.9 \times 13.33 \times 12)^2} = 3{,}686 \text{ kips}$$ *Eq. (10-10)*

For each of the exterior columns A2, A3, F2, and F3, (i.e., the columns with two beams framing into them in the direction of analysis):

$$\psi_A = \frac{355 + 419}{2 \times 70} = 5.5$$

$$\psi_B = 1.0$$

From the alignment chart, k = 1.75.

$$P_c = \frac{\pi^2 \times 34.5 \times 10^6}{(1.75 \times 13.33 \times 12)^2} = 4{,}345 \text{ kips}$$ *Eq. (10-10)*

For each of the 8 interior columns:

$$I_{col} = 0.7\left(\frac{24^4}{12}\right) = 19{,}354 \text{ in.}^4$$ *10.11.1*

For the column below level 2:

$$\frac{E_cI}{\ell_c} = \frac{4{,}415 \times 19{,}354}{(15 \times 12) - 10} = 503 \times 10^3 \text{ in.-kips}$$

For the column above level 2:

$$\frac{E_cI}{\ell_c} = \frac{4{,}415 \times 19{,}354}{12 \times 12} = 593 \times 10^3 \text{ in.-kips}$$

$$\psi_A = \frac{503 + 593}{2 \times 70} = 7.8$$

$$\psi_B = 1.0$$

From the alignment chart, k = 1.82.

$$EI = 0.4 \times 4{,}415 \times \frac{24^2}{12} = 48.8 \times 10^6 \text{ in.-kips}$$

Eq. (10-12)

$$P_c = \frac{\pi^2 EI}{(k\ell_u)^2} = \frac{\pi^2 \times 48.8 \times 10^6}{(1.82 \times 13.33 \times 12)^2} = 5{,}683 \text{ kips}$$

Therefore,

$\Sigma P_c = 12(3{,}686) + 4(4{,}345) + 8(5{,}683) = 107{,}076$ kips

The following table summarizes magnified moment computations for column C1 using 10.13.4.3 for all load conditions. The table is followed by detailed calculations for combinations no. 4 and 5 to illustrate the procedure.

No.	Load Combination	ΣP_u (kips)	δ_s (in.)	M_{2ns} (ft-kips)	M_{2s} (ft-kips)	M_2 (ft-kips)
1	1.4D	25,053	---	48.7	---	48.7
2	1.2D + 1.6L + 1.6L_r	24,795	---	66.4	---	66.4
3	1.2D + 0.5L + 1.6L_r	22,903	---	49.5	---	49.5
4	1.2D + 1.6L_r + 0.8W	21,908	1.38	21.1	110.4	173.5
5	1.2D + 1.6L_r - 0.8W	21,908	1.38	21.1	-110.4	-131.3
6	1.2D + 0.5L + 0.5L_r + 1.6W	22,605	1.39	25.0	220.8	331.9
7	1.2D + 0.5L + 0.5L_r - 1.6W	22,605	1.39	25.0	-220.8	-281.9
8	0.9D + 1.6W	16,106	1.25	15.8	220.8	292.0
9	0.9D - 1.6W	16,106	1.25	15.8	-220.8	-260.3

For load combinations No. 4 and 5:

$U = 1.2D + 1.6L_r \pm 0.8W$

$$\delta_s = \frac{1}{1-\dfrac{\Sigma P_u}{0.75\Sigma P_c}} = \frac{1}{1-\dfrac{21{,}908}{0.75 \times 107{,}076}} = 1.38$$

- For sidesway from north to south (load combination no. 4):

  $\delta_s M_{2s} = 1.38 \times 110.4 = 152.4$ ft-kips

  $M_2 = 21.1 + 152.4 = 173.5$ ft-kips

  $P_u = 722.0$ kips

- For sidesway from south to north (load combination no. 5):

  $\delta_s M_{2s} = 1.38 \times (-110.4) = -152.4$ ft-kips

  $M_2 = 21.1 - 152.4 = -131.3$ ft-kips

  $P_u = 799.3$ kips

A summary of the magnified moments for column C1 for all load combinations is provided in the following table.

No.	Load Combination	P_u (kips)	10.13.4.2		10.13.4.3	
			δ_s	M_2 (ft-kips)	δ_s	M_2 (ft-kips)
1	1.4D	871.4	– – –	48.7	– – –	48.7
2	1.2D + 1.6L + 0.5L$_r$	869.4	– – –	66.4	– – –	66.4
3	1.2D + 0.5L + 1.6L$_r$	797.6	– – –	49.5	– – –	49.5
4	1.2D + 1.6L$_r$ + 0.8W	722.0	1.14	147.0	1.38	173.5
5	1.2D + 1.6L$_r$ - 0.8W	799.3	1.14	-104.8	1.38	-131.3
6	1.2D + 0.5L + 0.5L$_r$ + 1.6W	710.9	1.14	276.7	1.39	331.9
7	1.2D + 0.5L + 0.5L$_r$ - 1.6W	865.4	1.14	-226.8	1.39	-281.9
8	0.9D + 1.6W	482.9	1.10	257.9	1.25	292.0
9	0.9D - 1.6W	637.4	1.10	-226.2	1.25	-260.3

d. Determine required reinforcement.

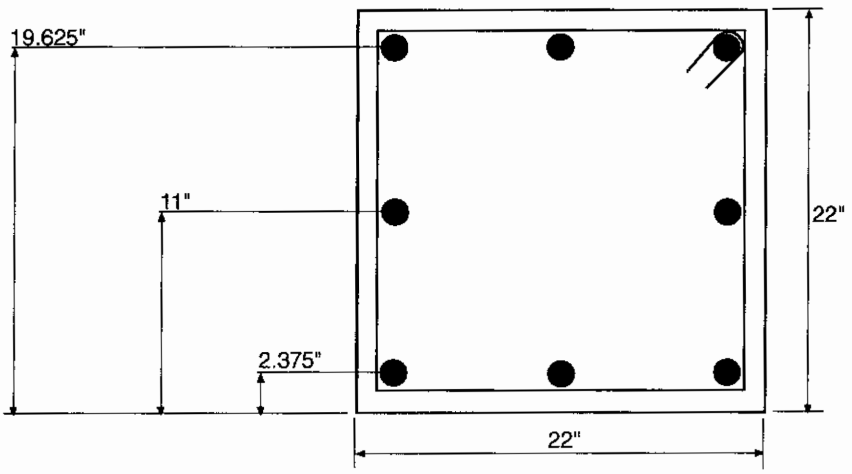
For the 22 × 22 in. column, try 8-No. 8 bars. Determine maximum allowable axial compressive force, $\phi P_{n,max}$:

$$\phi P_{n,max} = 0.80\phi\left[0.85f'_c\left(A_g - A_{st}\right) + f_y A_{st}\right]$$ *Eq. (10-2)*

$$= (0.80 \times 0.65)[(0.85 \times 6)(22^2 - 6.32) + (60 \times 6.32)]$$

$$= 1{,}464.0 \text{ kips} > \text{maximum } P_u = 871.4 \text{ kips O.K.}$$

The following table contains results from a strain compatibility analysis, where compressive strains are taken as positive (see Parts 6 and 7). Use $M_u = M_2$ from the approximate $P - \Delta$ method in 10.13.4.2.



No.	P_u (kips)	M_u (ft-kips)	c (in.)	ε_t	ϕ	ϕP_n (kips)	ϕM_n (ft-kips)
1	871.4	48.7	14.85	-0.00096	0.65	871.4	459.4
2	869.4	66.4	14.82	-0.00097	0.65	869.4	459.7
3	797.6	49.5	13.75	-0.00128	0.65	797.6	468.2
4	722.0	147.0	12.75	-0.00162	0.65	722.0	474.1
5	799.3	-104.8	13.78	-0.00127	0.65	799.3	468.0
6	710.9	276.7	12.61	-0.00167	0.65	710.9	474.8
7	865.4	-226.8	14.76	-0.00099	0.65	865.4	460.2
8	482.9	257.9	7.36	-0.00500	0.90	482.9	557.2
9	637.4	-226.2	11.68	-0.00204	0.65	637.4	478.8

Therefore, since $\phi M_n > M_u$ for all $\phi P_n = P_u$, use a 22 × 22 in. column with 8-No. 8 bars (ρ_g = 1.3%). The same reinforcement is also adequate for the load combinations from the magnified moment method of 10.13.4.3.

4. Design of column C2

a. Determine if slenderness effects must be considered.

In part 3(c), k was determined to be 1.82 for the interior columns. Therefore,

$$\frac{k\ell_u}{r} = \frac{1.82 \times 13.33 \times 12}{0.3 \times 24} = 40.4 > 22$$ *10.13.2*

Slenderness effects must be considered.

b. Determine total moment M_2 (including slenderness effects) and the design load combinations, using the approximate analysis of 10.13.4.2.

The following table summarizes magnified moment computation for column C2 for all load combinations, followed by detailed calculations for combinations no. 4 and 5 to illustrate the procedure.

No.	Load Combination	ΣP_u (kips)	Δ_o (in)	V_{us} (kips)	Q	δ_s	M_{2ns} (ft-kips)	M_{2s} (ft-kips)	M_2 (ft-kips)
1	1.4D	25,053	-	-	-	-	2.8	-	2.8
2	1.2D+1.6L+0.5Lr	24,795	-	-	-	-	27.4	-	27.4
3	1.2D+0.5L+1.6Lr	22,903	-	-	-	-	10.2	-	10.2
4	1.2D+1.6Lr+0.8W	21,908	0.28	302.6	0.12	1.14	-1.2	164.0	185.0
5	1.2D+1.6Lr-0.8W	21,908	0.28	302.6	0.12	1.14	-1.2	-164.0	-187.4
6	1.2D+0.5L+0.5Lr+1.6W	22,605	0.45	484.2	0.12	1.14	-5.1	328.0	368.9
7	1.2D+0.5L+0.5Lr-1.6W	22,605	0.45	484.2	0.12	1.14	-5.1	-328.0	-379.1
8	0.9D+1.6W	16,106	0.45	484.2	0.09	1.10	-0.9	328.0	358.6
9	0.9D-1.6W	16,106	0.45	484.2	0.09	1.10	-0.9	-328.0	-360.4

$$M_2 = M_{2ns} + M_{2s}$$ *Eq. (10-16)*

$$\delta_s M_{2s} = \frac{M_{2s}}{1-Q} \geq M_{2s}$$ *Eq. (10-17)*

For load combinations no. 4 and 5:

$U = 1.2D + 1.6L_r \pm 0.8W$

From part 3(b), δ_s was determined to be 1.14.

- For sidesway from north to south (load combination no. 4):

  $M_{2s} = 0.8 \times 205.0 = 164.0$ ft-kips

  $M_{2ns} = 1.2(-1.0) + 1.6 \times 0 = 1.2$ ft-kips

  $\delta_s M_{2s} = 1.14 \times 164 = 187.0$ ft-kips

  $M_2 = M_{2ns} + \delta_s M_{2s} = -1.2 + 187.0 = 185.8$ ft-kips

  $P_u = 1{,}332.6$ kips

- For sidesway from south to north (load combination no. 5):

  $\delta_s M_{2s} = 1.14 \times (-164) = -187.0$ ft-kips

  $M_2 = -1.2 - 187.0 = -188.2$ ft-kips

  $P_u = 1{,}333.0$ kips

c. For comparison purposes, recompute $\delta_s M_{2s}$ using the magnified moment method outlined in 10.13.4.3. Use the values of ΣP_u, ΣP_c, and δ_s computed in part 3(c).

No.	Load Combination	ΣP_u (kips)	δ_s (in.)	M_{2ns} (ft-kips)	M_{2s} (ft-kips)	M_2 (ft-kips)
1	1.4D	25,053	– – –	-2.8	– – –	-2.8
2	1.2D + 1.6L + 0.5L$_r$	24,795	– – –	-27.4	– – –	-27.4
3	1.2D + 0.5L + 1.6L$_r$	22,903	– – –	-10.2	– – –	-10.2
4	1.2D + 1.6L$_r$ + 0.8W	21,908	1.38	-1.2	164.0	225.1
5	1.2D + 1.6L$_r$ - 0.8W	21,908	1.38	-1.2	-164.0	-227.5
6	1.2D + 0.5L + 0.5L$_r$ + 1.6W	22,605	1.39	-5.1	328.0	451.4
7	1.2D + 0.5L + 0.5L$_r$ - 1.6W	22,605	1.39	-5.1	-328.0	-461.6
8	0.9D + 1.6W	16,106	1.25	-0.9	328.0	409.4
9	0.9D - 1.6W	16,106	1.25	-0.9	-328.0	-411.2

$U = 1.2D + 1.6L_r \pm 0.8W$

$\delta_s = 1.38$ from part 3(c)

- For sidesway from north to south (load combination no. 4):

$\delta_s M_{2s} = 1.38 \times 164.0 = 226.3$ ft-kips

$M_2 = -1.2 + 226.3 = 225.1$ ft-kips

$P_u = 1{,}332.6$ kips

- For sidesway from south to north (load combination no. 5):

  $\delta_s M_{2s} = 1.38 \times (-164.0) = -226.3$ ft-kips

  $M_2 = -1.2 - 226.3 = -227.5$ ft-kips

  $P_u = 1{,}333.0$ kips

A summary of the magnified moments for column C2 under all load combinations is provided in the following table.

No.	Load Combination	P_u (kips)	10.13.4.2 δ_s	10.13.4.2 M_2 (ft-kips)	10.13.4.3 δ_s	10.13.4.3 M_2 (ft-kips)
1	1.4D	1,522.6	---	-2.8	---	-2.8
2	1.2D + 1.6L + 0.5L_r	1,529.0	---	-27.4	---	-27.4
3	1.2D + 0.5L + 1.6L_r	1,400.1	---	-10.2	---	-10.2
4	1.2D + 1.6L_r + 0.8W	1,332.6	1.14	185.8	1.38	225.1
5	1.2D + 1.6L_r - 0.8W	1,333.0	1.14	-188.2	1.38	-227.5
6	1.2D + 0.5L + 0.5L_r + 1.6W	1,380.5	1.14	368.8	1.39	451.4
7	1.2D + 0.5L + 0.5L_r - 1.6W	1,381.5	1.14	-379.0	1.39	-461.6
8	0.9D + 1.6W	978.4	1.10	358.6	1.25	409.4
9	0.9D - 1.6W	979.3	1.10	-360.4	1.25	-411.2

d. Determine required reinforcement.

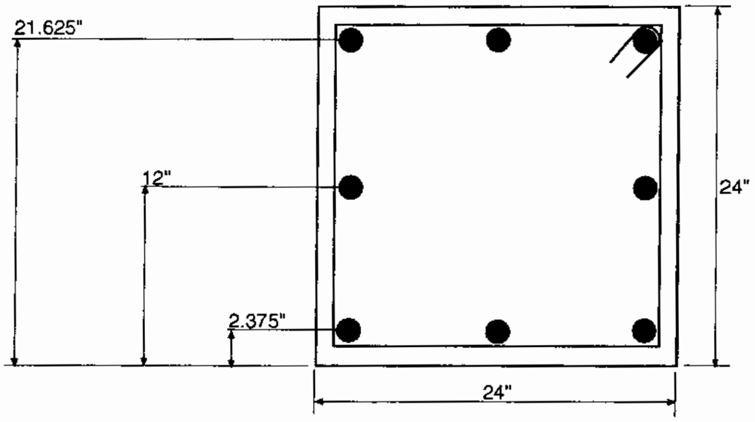
For the 24 × 24 in. column, try 8-No. 8 bars. Determine maximum allowable axial compressive force, $\phi P_{n,max}$:

$$\phi P_{n,max} = 0.80\phi\left[0.85f'_c\left(A_g - A_{st}\right) + f_y A_{st}\right]$$ *Eq. (10-2)*

$$= (0.80 \times 0.65)[(0.85 \times 6)(24^2 - 6.32) + (60 \times 6.32)]$$

$$= 1{,}708 \text{ kips} > \text{maximum } P_u = 1{,}529.0 \text{ kips O.K.}$$

The following table contains results from a strain compatibility analysis, where compressive strains are taken as positive (see Parts 6 and 7). Use $M_u = M_2$ from the approximate $P-\Delta$ method in 10.13.4.2.



No.	P_u (kips)	M_u (ft-kips)	c (in.)	ε_t	ϕ	ϕP_n (kips)	ϕM_n (ft-kips)
1	1,522.6	-2.8	23.30	0.00022	0.65	1,522.6	438.1
2	1,529.0	-27.4	23.39	0.00023	0.65	1,529.0	435.3
3	1,400.1	-10.2	21.49	-0.00002	0.65	1,400.1	489.7
4	1,332.6	185.8	20.50	-0.00016	0.65	1,332.6	513.3
5	1,333.0	-188.2	20.51	-0.00016	0.65	1,333.0	513.1
6	1,380.5	368.8	21.20	-0.00006	0.65	1,380.5	496.9
7	1,381.5	-379.0	21.22	-0.00005	0.65	1,381.5	496.4
8	978.4	358.6	15.52	-0.00118	0.65	978.4	587.1
9	979.3	-360.4	15.46	-0.00120	0.65	979.3	587.5

Therefore, since $\phi M_n > M_u$ for all $\phi P_n = P_u$, use a 24 × 24 in. column with 8-No. 8 bars ($\rho_g = 1.1\%$). The same reinforcement is also adequate for the load combinations from the magnified moment method of 10.13.4.3. Figure 11-16 obtained from pcaColumn[11.2] shows the design strength diagram for Column C2.

5. Determine if the maximum moment occurs at the end of the member. *10.13.5*

For column C1:

$$\frac{\ell_u}{r} = \frac{13.33 \times 12}{0.3 \times 22} = 24.2 < \frac{35}{\sqrt{\dfrac{871.4}{6 \times 22^2}}} = 63.9$$ *Eq. (10-19)*

For column C2:

$$\frac{\ell_u}{r} = \frac{13.33 \times 12}{0.3 \times 24} = 22.2 < \frac{35}{\sqrt{\dfrac{1,529.0}{6 \times 24^2}}} = 52.6$$

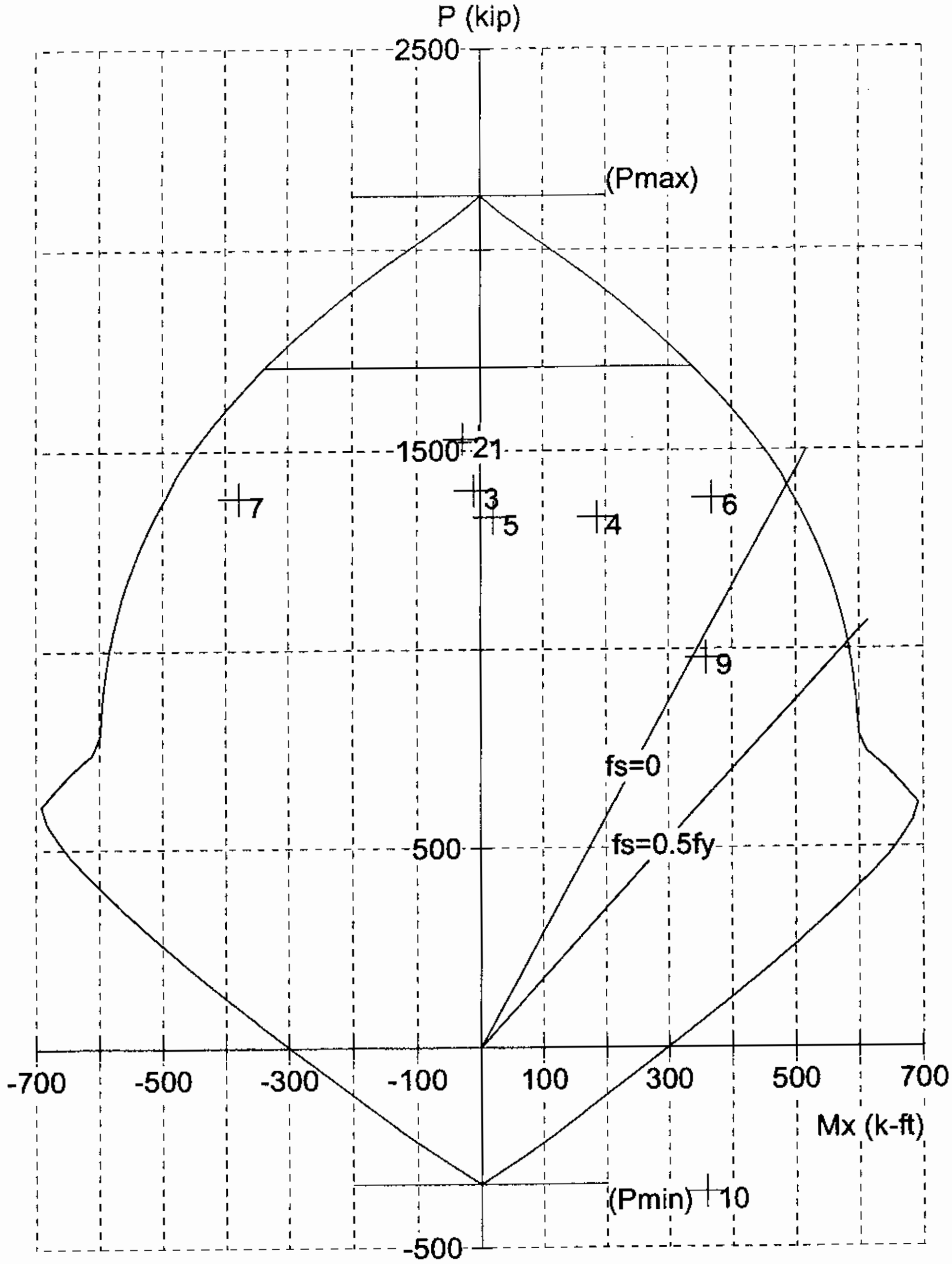


Figure 11-16 Design Strength Interaction Diagram for Column C2

Therefore, for columns C1 and C2, the maximum moment occurs at one of the ends, and the total moment M_2 does not have to be further magnified by δ_{ns}.

6. Check for sidesway instability of the structure. *10.13.6*

   a. When using 10.13.4.2 to compute $\delta_s M_s$, the value of Q evaluated using factored gravity loads shall not exceed 0.60. Note that for stability checks, all moments of inertia must be divided by $(1 + \beta_d)$ 10.11.1 where, for this story

$$\beta_d = \frac{\text{Maximum factored sustained axial load}}{\text{Maximum factored axial load}}$$

$$= \frac{1.4P_D}{P_u} = \frac{1.4 \times 17{,}895}{1.4 \times 17{,}895} = 1.0$$

$$1 + \beta_d = 2.0$$

Dividing all of the moments of inertia by $(1 + \beta_d)$ is equivalent to increasing the deflections, and consequently Q, by $(1 + \beta_d)$. Thus, at the second floor level,

$$Q = 2 \times 0.12 = 0.24 < 0.60$$

Therefore, the structure is stable at this level. In fact, computing the modified Q at each floor level shows that the entire structure is stable.

   b. When using 10.13.4.3 to compute $\delta_s M_s$, the value of δ_s computed using ΣP_u and ΣP_c corresponding to the factored dead and live loads shall be positive and shall not exceed 2.5. For the stability check, the values of EI must be divided by $(1 + \beta_d)$. Thus, the values of P_c must be recomputed considering the effects of β_d.

$$\Sigma P_c = \frac{107{,}076}{1 + 1} = 53{,}532 \text{ kips}$$

$$\text{and } \delta_s = \frac{1}{1 - \dfrac{(1.4 \times 17{,}895) + (1.7 \times 2{,}261)}{0.75 \times 53{,}532}} = 2.66 > 2.5$$

The structure is unstable when the magnified moment method of 10.13.4.3 is used.

Blank

12

Shear

UPDATE FOR THE '05 CODE

In the 2005 edition of the code, changes regarding shear are mostly editorial and notation related. Also, 11.5.3 and 17.5.2 have been inserted to clarify the definition of d for prestressed members.

GENERAL CONSIDERATIONS

The relatively abrupt nature of a failure in shear, as compared to a ductile flexural failure, makes it desirable to design members so that strength in shear is relatively equal to, or greater than, strength in flexure. To ensure that a ductile flexural failure precedes a shear failure, the code (1) limits the minimum and maximum amount of longitudinal reinforcement and (2) requires a minimum amount of shear reinforcement in all flexural members if the factored shear force V_u exceeds one-half of the shear strength provided by the concrete, ($V_u > 0.5\phi V_c$), except for certain types of construction (11.5.5.1), (3) specifies a lower strength reduction for shear ($\phi = 0.75$) than for tension-controlled section under flexure ($\phi = 0.90$).

The determination of the amount of shear reinforcement is based on a modified form of the truss analogy. The truss analogy assumes that shear reinforcement resists the total transverse shear. Considerable research has indicated that shear strength provided by concrete V_c can be assumed equal to the shear causing inclined cracking; therefore, shear reinforcement need be designed to carry only the excess shear.

Only shear design for nonprestressed members with clear-span-to-overall-depth ratios greater than 4 is considered in Part 12. Also included is horizontal shear design in composite concrete flexural members, which is covered separately in the second half of Part 12. Shear design for deep flexural members, which have clear-span-to-overall-depth ratios less than 4, is presented in Part 17. Shear design of prestressed members is discussed in Part 25. The alternate shear design method of Appendix A, Strut-and-Tie Models, is discussed in Part 32.

11.1 SHEAR STRENGTH

Design provisions for shear are presented in terms of shear forces (rather than stresses) to be compatible with the other design conditions for the strength design method, which are expressed in terms of loads, moments, and forces.

Accordingly, shear is expressed in terms of the factored shear force V_u, using the basic shear strength requirement:

Design shear strength ≥ Required shear strength

$$\phi V_n \geq V_u \qquad \textit{Eq. (11-1)}$$

The nominal shear strength V_n is computed by:

$$V_n = V_c + V_s \qquad \textit{Eq. (11-2)}$$

where V_c is the nominal shear strength provided by concrete and V_s is the nominal shear strength provided by shear reinforcement.

Equation (11-2) can be substituted into Eq. (11-1) to obtain:

$$\phi V_c + \phi V_s \geq V_u$$

The required shear strength at any section is computed using Eqs. (11-1) and (11-2), where the factored shear force V_u is obtained by applying the load factors specified in 9.2. The strength reduction factor, $\phi = 0.75$, is specified in 9.3.2.3.

11.1.1.1 Web Openings

Often it is necessary to modify structural components of buildings to accommodate necessary mechanical and electrical service systems. Passing these services through openings in the webs of floor beams within the floor-ceiling sandwich eliminates a significant amount of dead space and results in a more economical design. However, the effect of the openings on the shear strength of the floor beams must be considered, especially when such openings are located in regions of high shear near supports. In 11.1.1.1, the code requires the designer to consider the effect of openings on the shear strength of members. Because of the many variables such as opening shape, size, and location along the span, specific design rules are not stated. However, references are given for design guidance in R11.1.1.1. Generally, it is desirable to provide additional vertical stirrups adjacent to both sides of a web opening, except for small isolated openings. The additional shear reinforcement can be proportioned to carry the total shear force at the section where an opening is located. Example 12.5 illustrates application of a design method recommended in Ref. 12.1.

11.1.2 Limit on $\sqrt{f'_c}$

Concrete shear strength equations presented in Chapter 11 of the Code are a function of $\sqrt{f'_c}$, and had been verified experimentally for members with concrete compressive strength up to 10,000 psi. Due to a lack of test data for members with $f'_c > 10{,}000$ psi, 11.1.2 limits the value of $\sqrt{f'_c}$ to 100 psi, except as allowed in 11.1.2.1.

Section 11.1.2 does not prohibit the use of concrete with $f'_c > 10{,}000$ psi; it merely directs the engineer not to count on any strength in excess of 10,000 psi when computing V_c, unless minimum shear reinforcement is provided in accordance with 11.1.2.1.

It should be noted that prior to the 2002 Code, minimum area of transverse reinforcement was independent of the concrete strength. However, tests indicated that an increase in the minimum amount of transverse reinforcement is required for members with high-strength concrete to prevent sudden shear failures when inclined cracking occurs. Thus, to account for this, minimum transverse reinforcement requirements are a function of $\sqrt{f'_c}$.

11.1.3 Computation of Maximum Factored Shear Force

Section 11.1.3 describes three conditions that shall be satisfied in order to compute the maximum factored shear force V_u in accordance with 11.1.3.1 for nonprestressed members:

1. Support reaction, in direction of applied shear force, introduces compression into the end regions of the member.
2. Loads are applied at or near the top of the member.
3. No concentrated load occurs between the face of the support and the location of the critical section, which is a distance d from the face of the support (11.1.3.1).

When the conditions of 11.1.3 are satisfied, sections along the length of the member located less than a distance d from the face of the support are permitted to be designed for the shear force V_u computed at a distance d from the face of the support. See Fig. 12-1 (a), (b), and (c) for examples of support conditions where 11.1.3 would be applicable.

Conditions where 11.1.3 cannot be applied include: (1) members framing into a supporting member in tension, see Fig. 12-1 (d); (2) members loaded near the bottom, see Fig. 12-1 (e); and (3) members subjected to an abrupt change in shear force between the face of the support and a distance d from the face of the support, see Fig. 12-1 (f). In all of these cases, the critical section for shear must be taken at the face of the support. Additionally, in the case of Fig. 12-1 (d), the shear within the connection must be investigated and special corner reinforcement should be provided.

One other support condition is noteworthy. For brackets and corbels, the shear at the face of the support V_u must be considered, as shown in Fig. 12-2. However, these elements are more appropriately designed for shear using the shear-friction provisions of 11.7. See Part 15 for design of brackets and corbels.

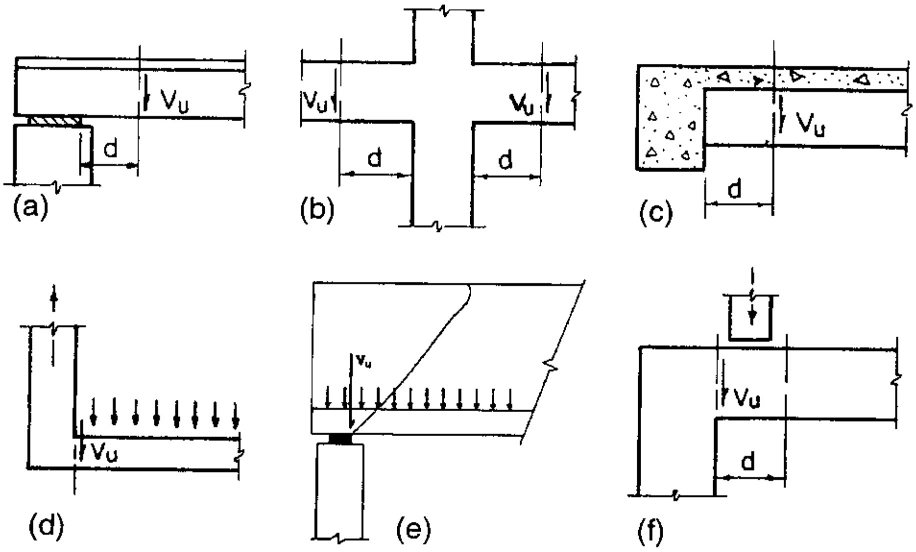


Figure 12-1 Typical Support Conditions for Locating Factored Shear Force V_u

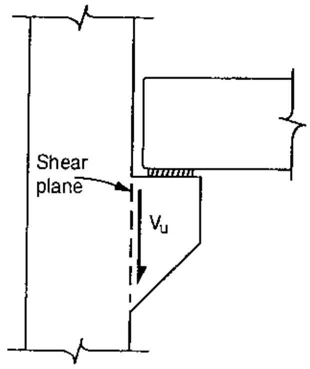


Figure 12-2 Critical Shear Plane for Brackets and Corbels

11.2 LIGHTWEIGHT CONCRETE

Since the shear strength of lightweight aggregate concrete may be less than that of normal weight concrete with equal compressive strength, adjustments in the value of V_c, as computed for normal weight concrete, are necessary.

Except for 11.5.5.3, 11.5.7.9, 11.6.3.1, 11.12.3.2, and 11.12.4.8, when average splitting tensile strength f_{ct} is specified, $f_{ct} / 6.7$ is substituted for $\sqrt{f'_c}$ in all equations of Chapter 11. However, the value of $f_{ct} / 6.7$ cannot be taken greater than $\sqrt{f'_c}$. When f_{ct} is not specified, $\sqrt{f'_c}$ is reduced using a multiplier of 0.75 for all-lightweight concrete or 0.85 for sand-lightweight concrete. Linear interpolation between these multipliers is allowed when partial sand replacement is used. Section 11.7.4.3 specifies the same multipliers for lightweight concrete.

11.3 SHEAR STRENGTH PROVIDED BY CONCRETE FOR NONPRESTRESSED MEMBERS

When computing the shear strength provided by concrete for members subject to shear and flexure only, designers have the option of using either the simplified equation, $V_c = 2\sqrt{f'_c}\, b_w d$ [Eq. (11-3)], or the more elaborate expression given by Eq. (11-5). In computing V_c from Eq. (11-5), it should be noted that V_u and M_u are the values which occur simultaneously at the section considered. A maximum value of 1.0 is prescribed for the ratio $V_u d/M_u$ to limit V_c near points of inflection where M_u is zero or very small.

For members subject to shear and flexure with axial compression, a simplified V_c expression is given in 11.3.1.2, with an optional more elaborate expression for V_c available in 11.3.2.2. For members subject to shear, flexure and significant axial tension, 11.3.1.3 requires that shear reinforcement must be provided to resist the total shear unless the more detailed analysis of 11.3.2.3 is performed. Note that N_u represents a tension force in Eq. (11-8) and is therefore taken to be negative.

No precise definition is given for "significant axial tension." If there is uncertainty about the magnitude of axial tension, it may be desirable to carry all applied shear by shear reinforcement.

Figure 12-3 shows the variation of shear strength provided by concrete, V_c as function of $\sqrt{f'_c}$, $V_u d/M_u$, and reinforcement ratio ρ_w.

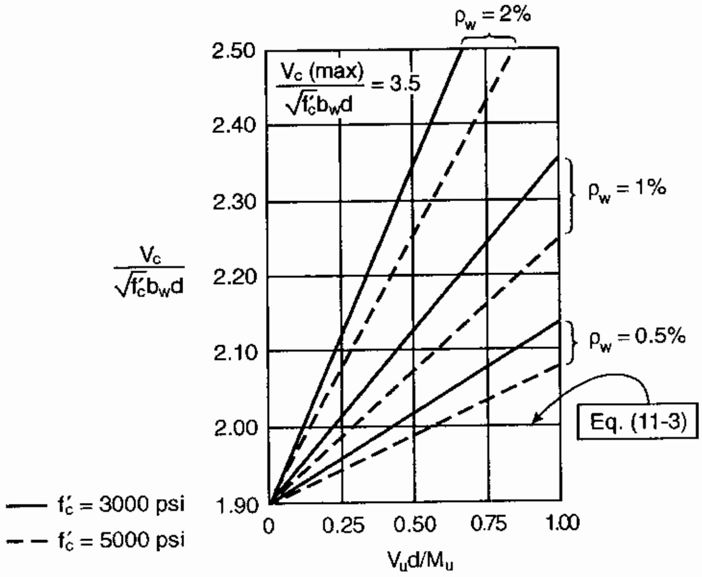


Figure 12-3 Variation of $V_c / \sqrt{f'_c}\, b_w d$ with f'_c, ρ_w, and $V_u d/M_u$ using Eq. (11-5)

Figure 12-4 shows the approximate range of values of V_c for sections under axial compression, as obtained from Eqs. (11-5) and (11-6). Values correspond to a 6 ×12 in. beam section with an effective depth of 10.8 in. The curves corresponding to the alternate expressions for V_c given by Eqs. (11-4) and (11-7), as well as that corresponding to Eq. (11-8) for members subject to axial tension, are also indicated.

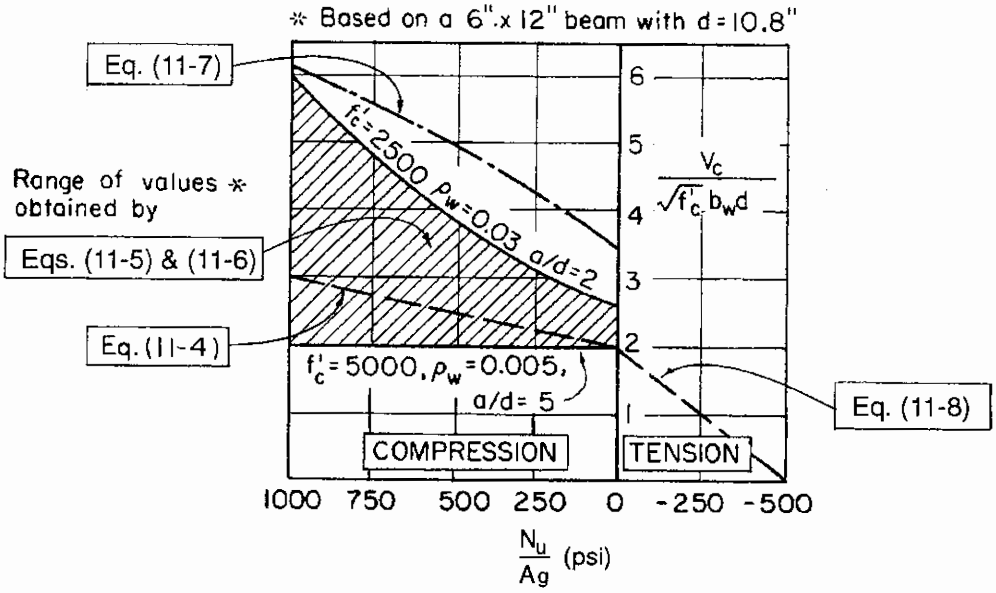


Figure 12-4 Comparison of Design Equations for Shear and Axial Load

Figure 12-5 shows the variation of V_c with N_u/A_g and f_c' for sections subject to axial compression, based on Eq. (11-4). For the range of N_u/A_g values shown, V_c varies from about 49% to 57% of the value of V_c as defined by Eq. (11-7).

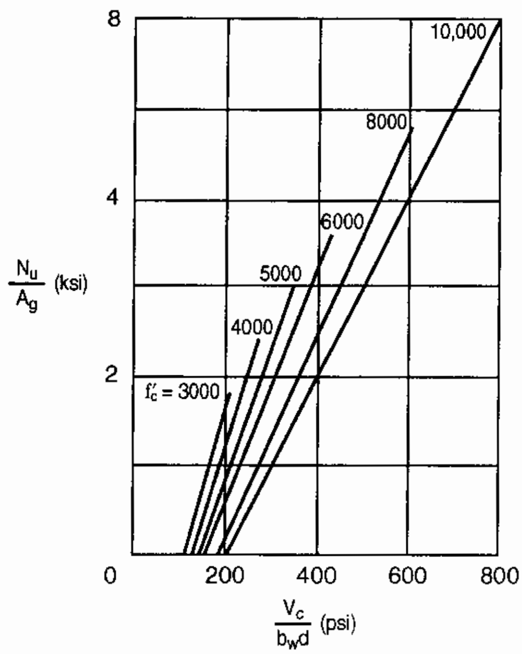


Figure 12-5 Variation of V_c/b_wd with f_c' and N_u/A_g using Eq. (11-4)

11.5 SHEAR STRENGTH PROVIDED BY SHEAR REINFORCEMENT

11.5.1 Types of Shear Reinforcement

Several types and arrangements of shear reinforcement permitted by 11.5.1.1 and 11.5.1.2 are illustrated in Fig. 12-6. Spirals, circular ties, or hoops are explicitly recognized as types of shear reinforcement starting with the 1999 code. Vertical stirrups are the most common type of shear reinforcement. Inclined stirrups and longitudinal bent bars are rarely used as they require special care during placement in the field.

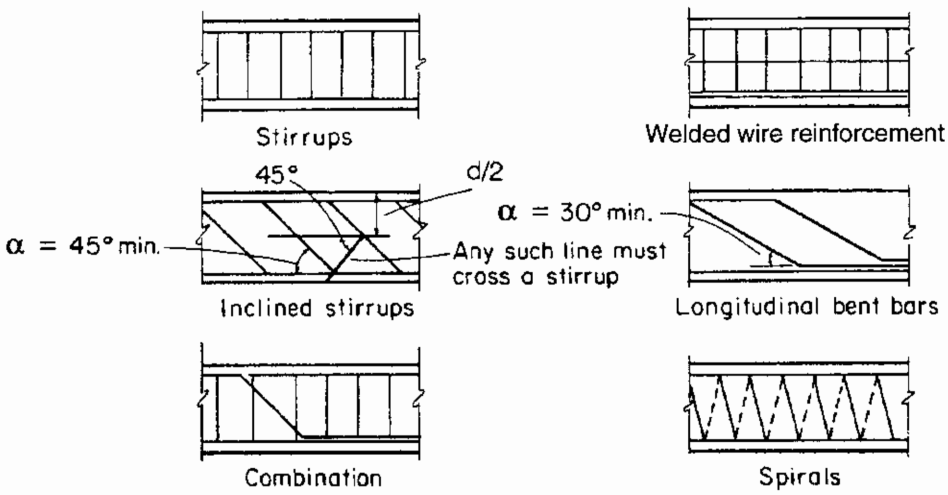


Figure 12-6 Types and Arrangements of Shear Reinforcement

11.5.4 Anchorage Details for Shear Reinforcement

To be fully effective, shear reinforcement must extend as close to full member depth as cover requirements and proximity of other reinforcement permit (12.13.1), and be anchored at both ends to develop the design yield strength of the shear reinforcement. The anchorage details prescribed in 12.13 are presumed to satisfy this development requirement.

11.5.5 Spacing Limits for Shear Reinforcement

Spacing of stirrups and welded wire reinforcement, placed perpendicular to axis of member, must not exceed one-half the effective depth of the member (d/2), nor 24 in. When the quantity $\phi V_s = (V_u - \phi V_c)$ exceeds $\phi 4\sqrt{f'_c}\, b_w d$, maximum spacing must be reduced by one-half to (d/4) or 12 in. Note also that the value of (ϕV_s) shall not exceed $\phi 8\sqrt{f'_c}\, b_w d$ (11.5.7.9). For situations where the required shear strength exceeds this limit, the member size or the strength of the concrete may be increased to provide additional shear strength provided by concrete.

11.5.6 Minimum Shear Reinforcement

When the factored shear force V_u exceeds one-half the shear strength provided by concrete ($V_u > \phi V_c/2$), a minimum amount of shear reinforcement must be provided in concrete flexural members, except for slabs and footings, joists defined by 8.11, and wide, shallow beams (11.5.5.1). When required, the minimum shear reinforcement for nonprestressed members is

$$A_{v,min} = 0.75\sqrt{f'_c}\,\frac{b_w s}{f_{yt}} \qquad \textit{Eq. (11-13)}$$

but not less than $\frac{50 b_w s}{f_{yt}}$

Minimum shear reinforcement is a function of the concrete compressive strength starting with the 2002 Code. Equation (11-13) provides a gradual increase in the minimum required $A_{v,min}$, while maintaining the previous minimum value of $50\ b_w s / f_{yt}$.

Note that spacing of minimum shear reinforcement must not exceed d/2 or 24 in.

11.5.7 Design of Shear Reinforcement

When the factored shear force V_u exceeds the shear strength provided by concrete, ϕV_c, shear reinforcement must be provided to carry the excess shear. The code provides an equation that defines the required shear strength V_s provided by reinforcement in terms of its area A_v, yield strength f_{yt}, and spacing s. [Eq. (11-15)]. The equation is based on a truss model with the inclination angle of compression diagonals equal to 45 degree.

To assure correct application of the strength reduction factor, ϕ, equations for directly computing required shear reinforcement A_v are developed below. For shear reinforcement placed perpendicular to the member axis, the following method may be used to determine the required area of shear reinforcement A_v, spaced at a distance s:

$$\phi V_n \geq V_u \qquad \textit{Eq. (11-1)}$$

where $$V_n = V_c + V_s \qquad \textit{Eq. (11-2)}$$

and $$V_s = \frac{A_v f_{yt} d}{s} \qquad \textit{Eq. (11-15)}$$

Substituting V_s into Eq. (11-2) and V_n into Eq. (11-1), the following equation is obtained:

$$\phi V_c + \frac{\phi A_v f_{yt} d}{s} \geq V_u$$

Solving for A_v,

$$A_v = \frac{(V_u - \phi V_c)\, s}{\phi f_{yt} d}$$

Similarly, when inclined stirrups are used as shear reinforcement,

$$A_v = \frac{(V_u - \phi V_c)\, s}{\phi f_{yt}\,(\sin\alpha + \cos\alpha)\, d}$$

where α is the angle between the inclined stirrup and longitudinal axis of member (see Fig. 12-8).

When shear reinforcement consists of a single bar or group of parallel bars, all bent-up at the same distance from the support,

$$A_v = \frac{(V_u - \phi V_c)}{f_y \sin\alpha}$$

where α is the angle between the bent-up portion and longitudinal axis of member, but not less than 30 degree (see Fig. 12-6). For this case, the quantity (V_u - ϕV_c) must not exceed $\phi 3\sqrt{f_c'} b_w d$.

Design Procedure for Shear Reinforcement

Design of a nonprestressed concrete beam for shear involves the following steps:

1. Determine maximum factored shear force V_u at critical sections of the member per 11.1.3 (see Fig. 12-1).
2. Determine shear strength provided by the concrete ϕV_c per Eq. (11-3): $\phi V_c = \phi 2\sqrt{f_c'} b_w d$ where $\phi = 0.75$ (9.3.2.3).
3. Compute V_u - ϕV_c at the critical section. If $V_u - \phi V_c > \phi 8\sqrt{f_c'} b_w d$, increase the size of the section or the concrete compressive strength.
4. Compute the distance from the support beyond which minimum shear reinforcement is required (i.e., where ($V_u = \phi V_c$), and the distance from the support beyond which the concrete can carry the total shear force (i.e., where $V_u = \phi V_c/2$).
5. Use Table 12-1 to determine the required area of vertical stirrups A_v or stirrup space s at a few controlling sections along the length of the member, which includes the critical sections.

   Where stirrups are required, it is usually more expedient to select a bar size and type (e.g., No. 3 U-stirrups (2 legs)) and determine the required spacing. Larger stirrup sizes at wider spacings are usually more cost effective than smaller stirrup sizes at closer spacings because the latter requires disproportionately high costs for fabrication and placement. Changing the stirrup spacing as few times as possible over the required length also results in cost savings. If possible, no more than three different stirrup spacings should be specified, with the first stirrup located 2 in. from the face of the support.

Table 12-1 Provisions for Shear Design

		$V_u \le \phi V_c/2$	$\phi V_c/2 < V_u \le \phi V_c$	$\phi V_c < V_u$
Required area of stirrups, A_v		none	$0.75\sqrt{f_c'}\dfrac{b_w s}{f_{yt}} \ge \dfrac{50 b_w s}{f_{yt}}$	$\dfrac{(V_u - \phi V_c)s}{\phi f_{yt}}$
Stirrup spacing, s	Required	—	$\dfrac{A_v f_{yt}}{0.75\sqrt{f_c'} b_w} \le \dfrac{A_v f_{yt}}{50 b_w}$	$\dfrac{\phi A_v f_{yt}}{V_u - \phi V_c}$
	Maximum	—	$d/2 \le 24$ in.	$d/2 \le 24$ in. for $(V_u - \phi V_c) \le \phi 4\sqrt{f_c'} b_w d$
				$d/2 \le 12$ in. for $\phi 4\sqrt{f_c'} b_w d < (V_u - \phi V_c) \le \phi 8\sqrt{f_c'} b_w d$

The shear strength requirements are illustrated in Fig. 12-7.

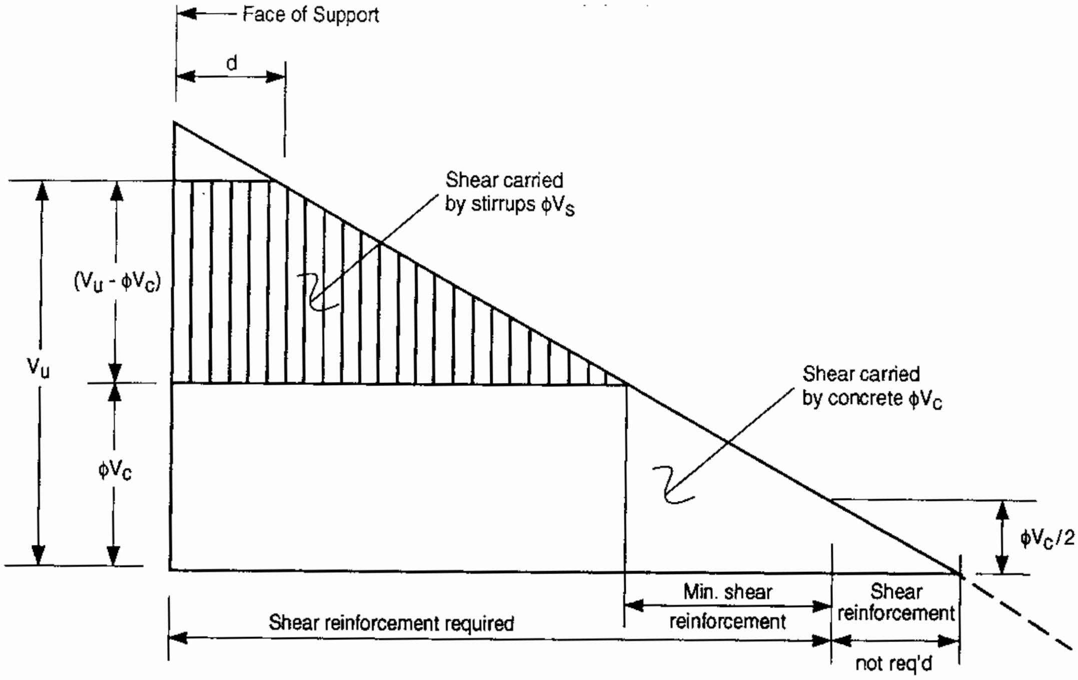


Figure 12-7 Shear Strength Requirements

The expression for shear strength provided by shear reinforcement ϕV_s can be assigned specific force values for a given stirrup size and strength of reinforcement. The selection and spacing of stirrups can be simplified if the spacing is expressed as a function of the effective depth d instead of numerical values. Practical limits of stirrup spacing generally vary from s = d/2 to s = d/4, since spacing closer than d/4 is not economical. With one intermediate spacing at d/3, a specific value of ϕV_s can be derived for each stirrup size and spacing as follows:

For vertical stirrups:

$$\phi V_s = \frac{\phi A_v f_{yt} d}{s} \qquad \textit{Eq. (11-7)}$$

Substituting d/n for s, where n = 2, 3, and 4

$$\phi V_s = \phi A_v f_{yt} n$$

Thus, for No. 3 U-stirrups @ s = d/2, f_{yt} = 60 ksi and ϕ = 0.75

$$\phi V_s = 0.75\,(2 \times 0.11)\,60 \times 2 = 19.8 \text{ kips, say 19 kips}$$

Values of ϕV_s given in Table 12-2 may be used to select shear reinforcement. Note that the ϕV_s values are independent of member size and concrete strength. Selection and spacing of stirrups using the design values for $\phi V_s = (V_u - \phi V_c)$ can be easily solved by numerical calculation or graphically. See Example 12.1.

Table 12-2 Shear Strength ϕV_s for Given Bar Sizes and Spacings

Spacing	Shear Strength ϕV_s (kips)					
	No. 3 U-Stirrups*		No. 4 U-Stirrups*		No. 5 U-Stirrups*	
	Grade 40	Grade 60	Grade 40	Grade 60	Grade 40	Grade 60
d/2	13	19	24	36	37	55
d/3	19	29	36	54	55	83
d/4	26	39	48	72	74	111

* *Stirrups with 2 legs (double values for 4 legs, etc.)*

CHAPTER 17 — COMPOSITE CONCRETE FLEXURAL MEMBERS

17.4 VERTICAL SHEAR STRENGTH

Section 17.4.1 of the Code permits the use of the entire composite flexural member to resist the design vertical shear as if the member were monolithically cast. Therefore, the requirements of Code Chapter 11 apply.

Section 17.4.3 permits the use of vertical shear reinforcement to serve as ties for horizontal shear reinforcement, provided that the vertical shear reinforcement is extended and anchored in accordance with applicable provisions.

17.5 HORIZONTAL SHEAR STRENGTH

In composite flexural members, horizontal shear forces are caused by the moment gradient resulting from vertical shear force. These horizontal shear forces act over the interface of interconnected elements that form the composite member.

Section 17.5.1 requires full transfer of the horizontal shear forces by friction at the contact surface, properly anchored ties, or both. Unless calculated in accordance with 17.5.4, the factored applied horizontal shear force $V_u \leq \phi V_{nh}$, where ϕV_{nh} is the horizontal shear strength (17.5.3).

The horizontal shear strength is $\phi V_{nh} = 80 b_v d$ for intentionally roughened contact surfaces without the use of ties (friction only), and for surfaces that are not intentionally roughened with the use of minimum ties provided in accordance with 17.6 (17.5.3.1 and 17.5.3.2). When ties per 17.6 are provided, and the contact surface is intentionally roughened to a full amplitude of approximately 1/4 in., the horizontal shear strength is:

$$\phi V_{nh} = (260 + 0.6\rho_v f_{yt})\lambda b_v d \leq 500 b_v d \quad (17.5.3.3).$$

The expression for V_{nh} in 17.5.3.3 accounts for the effect of the quantity of reinforcement crossing the interface by including ρ_v, which is the ratio of tie reinforcement area to area of contact surface, or $\rho_v = A_v/b_v s$. It also incorporates the correction factor λ to account for lightweight aggregate concrete per 11.7.4.3. It should also be noted that for concrete compressive strength $f_c' \leq 4444$ psi, the minimum tie reinforcement per Eq. (11-13) is $\rho_v f_{yt} = 50$ psi; substituting this into the above expression, $V_{nh} = 290\lambda b_v d$. The upper limit of $500 b_v d$ corresponds to $\rho_v f_{yt} = 400$ psi in the case of normal weight concrete (i.e., $\lambda = 1$).

When in computing the horizontal shear strength of a composite flexural member, the following apply:

1. When $V_u > \phi(500 b_v d)$, the shear friction method of 11.7.4 must be used (17.5.2.4). Refer to Part 14 for further details on the application of 11.7.4.
2. No distinction shall be made between shored or unshored members (17.2.4). Tests have indicated that the strength of a composite member is the same whether or not the first element cast is shored or not.
3. Composite members must meet the appropriate requirements for deflection control per 9.5.6.
4. The contact surface shall be clean and free of laitance. Intentionally roughened surface may be achieved by scoring the surface with a stiff bristled broom. Heavy raking or grooving of the surface may be sufficient to achieve "full 1/4 in. amplitude."
5. The effective depth d is defined as the distance from the extreme compression fiber for the entire composite section to the centroid of the tension reinforcement. For prestressed member, the effective depth shall not be taken less than 0.80 h (17.5.2).

The code also presents an alternative method for horizontal shear design in 17.5.4. The horizontal shear force that must be transferred across the interface between parts of a composite member is taken to be the change in internal compressive or tensile force, parallel to the interface, in any segment of a member. When this method is used, the limits of 17.5.3.1 through 17.5.3.4 apply, with the contact area A_c substituted for the quantity $b_v d$ in the expressions. Section 17.5.4.1 also requires that the reinforcement be distributed to approximately reflect the variation in shear force along the member. This requirement emphasizes the difference between the design of composite members on concrete and on steel. Slip between the steel beam and composite concrete slab at maximum strength is large, which permits redistribution of the shear force along the member. In concrete members with a composite slab, the slip at maximum strength is small and redistribution of shear resistance along the member is limited. Therefore, distribution of horizontal shear reinforcement must be based on the computed distribution of factored horizontal shear in concrete composite flexural members.

17.6 TIES FOR HORIZONTAL SHEAR

According to 17.6.3, ties are required to be "fully anchored" into interconnected elements "in accordance with 12.13." Figure 12-8 shows some tie details that have been used successfully in testing and design practice. Figure 12-8(a) shows an extended stirrup detail used in tests of Ref. 12.3. Use of an embedded "hairpin" tie, as illustrated in Fig. 12-8(b), is common practice in the precast, prestressed concrete industry. Many precast products are manufactured in such a way that it is difficult to position tie reinforcement for horizontal shear before concrete is placed. Accordingly, the ties are embedded in the plastic concrete as permitted by 16.7.1.

Shear reinforcement that extends from previously-cast concrete and is adequately anchored into the composite portion of a member (Fig. 12-8(c)) may be used as reinforcement (ties) to resist horizontal shear (17.4.3). Therefore, this reinforcement may be used to satisfy requirements for both vertical and horizontal shear.

Example 12.6 illustrates design for horizontal shear.

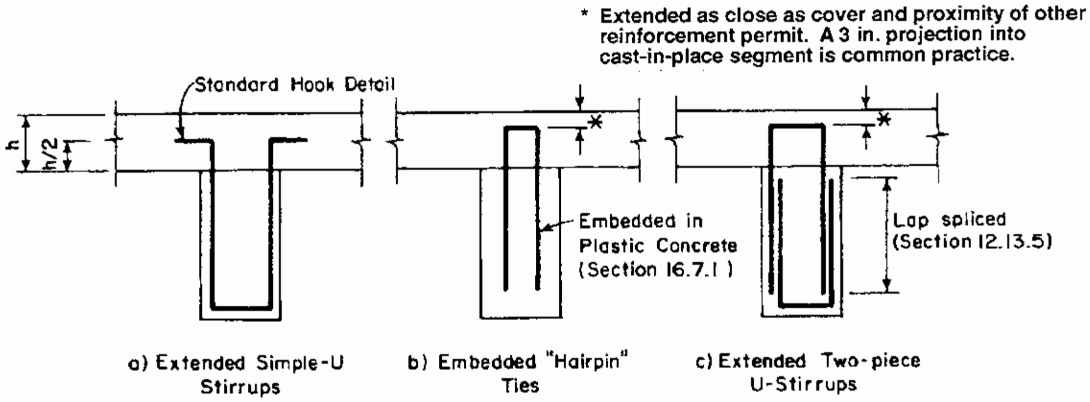


Figure 12-8 Ties for Horizontal Shear

REFERENCES

12.1 Barney, G.B.; Corley, W.G.; Hanson, J.M.; and Parmelee, R.A., "Behavior and Design of Prestressed Concrete Beams with Large Web Openings," *PCI Journal*, V. 22, No. 6, November-December 1977, pp. 32-61. Also, *Research and Development Bulletin* RD054D, Portland Cement Association, Skokie, IL.

12.2 Hanson, N.W., *Precast-Prestressed Concrete Bridges 2. Horizontal Shear Connections*, Development Department Bulletin D35, Portland Cement Association, Skokie, IL, 1960.

12.3 Roller, J.J., and Russell, H.G., "Shear Strength of High Strength Concrete Beams with Web Reinforcement," *ACI Structural Journal*, Vol. 87, No. 2, March-April 1990, pp. 191-198.

Example 12.1—Design for Shear - Members Subject to Shear and Flexure Only

Determine required size and spacing of vertical U-stirrups for a 30-foot span, simply supported beam.

b_w = 13 in.
d = 20 in.
f'_c = 3000 psi
f_{yt} = 40,000 psi
w_u = 4.5 kips/ft

Calculations and Discussion	Code Reference
For the purpose of this example, the live load will be assumed to be present on the full span, so that design shear at centerline of span is zero. (A design shear greater than zero at midspan is obtained by considering partial live loading of the span.) Using design procedure for shear reinforcement outlined in this part:	
1. Determine factored shear forces	
@ support: $V_u = 4.5\,(15) = 67.5$ kips	
@ distance d from support:	
$V_u = 67.5 - 4.5\,(20/12) = 60$ kips	*11.1.3.1*
2. Determine shear strength provided by concrete	
$\phi V_c = \phi 2\sqrt{f'_c}\, b_w d$	*Eq. (11-3)*
$\phi = 0.75$	*9.3.2.3*
$\phi V_c = 0.75\,(2)\,\sqrt{3000} \times 13 \times 20 / 1000 = 21.4$ kips	
$V_u = 60 \text{ kips} > \phi V_c = 21.4$ kips	
Therefore, shear reinforcement is required.	*11.1.1*
3. Compute $V_u - \phi V_c$ at critical section.	
$V_u - \phi V_c = 60 - 21.4 = 38.6 \text{ kips} < \phi 8\sqrt{f'_c}\, b_w d = 85.4$ kips O.K.	*11.5.7.9*

4. Determine distance x_c from support beyond which minimum shear reinforcement is required ($V_u = \phi V_c$):

$$x_c = \frac{V_u \text{ @ support} - \phi V_c}{w_u} = \frac{67.5 - 21.4}{4.5} = 10.2 \text{ ft}$$

Determine distance x_m from support beyond which concrete can carry total shear force ($V_u = \phi V_c / 2$):

$$x_m = \frac{V_u \text{ @ support} - (\phi V_c / 2)}{w_u} = \frac{67.5 - (21.4/2)}{4.5} = 12.6 \text{ ft}$$

5. Use Table 12-1 to determine required spacing of vertical U-stirrups.

At critical section, $V_u = 60$ kips > $\phi V_c = 21.4$ kips

$$s\text{ (req'd)} = \frac{\phi A_v f_{yt} d}{V_u - \phi V_c}$$ *Eq. (11-15)*

Assuming No. 4 U-stirrups ($A_v = 0.40$ in.^{2}),

$$s\text{ (req'd)} = \frac{0.75 \times 0.40 \times 40 \times 20}{38.6} = 6.2 \text{ in.}$$

Check maximum permissible spacing of stirrups:

$s\text{ (max)} \le d/2 = 20/2 = 10$ in. (governs) *11.5.5.1*

≤ 24 in. since $V_u - \phi V_c = 38.6$ kips $< \phi 4\sqrt{f'_c}\, b_w d = 42.7$ kips

Maximum stirrup spacing based on minimum shear reinforcement:

$$s\text{ (max)} \le \frac{A_v f_{yt}}{0.75\sqrt{f'_c} b_w} = \frac{0.4 \times 40{,}000}{0.75\sqrt{3{,}000}(13)} = 30 \text{ in.}$$ *11.5.6.3*

$$\le \frac{A_v f_{yt}}{50 b_w} = \frac{0.4 \times 40{,}000}{50 \times 13} = 24.6 \text{ in.}$$

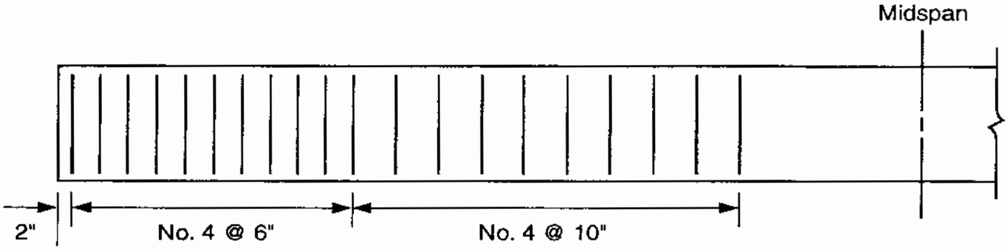
Determine distance x from support beyond which 10 in. stirrup spacing may be used:

$$10 = \frac{0.75 \times 0.4 \times 40 \times 20}{V_u - 21.4}$$

$V_u - 21.4 = 24$ kips or $V_u = 24 + 21.4 = 45.4$ kips

$$x = \frac{67.5 - 45.4}{4.5} = 4.9 \text{ ft}$$

Stirrup spacing using No. 4 U-stirrups:



6. As an alternate procedure, use simplified method presented in Table 12-2 to determine stirrup size and spacing.

At critical section,

$\phi V_s = V_u - \phi V_c = 60 - 21.4 = 38.6$ kips

From Table 12-2 for Grade 40 stirrups:

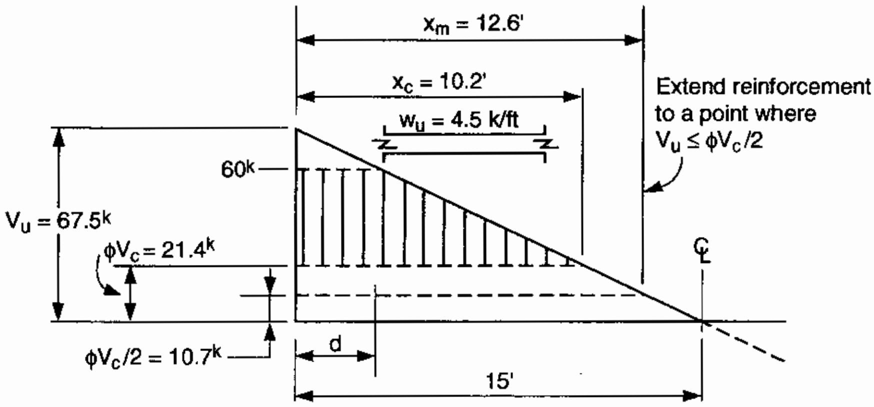
No. 4 U-stirrups @ d/4 provides $\phi V_s = 48$ kips

No. 4 U-stirrups @ d/3 provides $\phi V_s = 36$ kips

By interpolation, No. 4 U-stirrups @ d/3.22 = 38.6 kips

Stirrup spacing = d/3.22 = 20/3.22 = 6.2 in.

Stirrup spacing along length of beam is determined as shown previously.



Example 12.2—Design for Shear - with Axial Tension

Determine required spacing of vertical U-stirrups for a beam subject to axial tension.

f'_c = 3600 psi (sand-lightweight concrete, f_{ct} not specified)
f_{yt} = 40,000 psi
M_d = 43.5 ft-kips
M_ℓ = 32.0 ft-kips
V_d = 12.8 kips
V_ℓ = 9.0 kips
N_d = -2.0 kips (tension)
N_ℓ = -15.2 kips (tension)

Calculations and Discussion	Code Reference

1. Determine factored loads — *9.2.1*

$M_u = 1.2\,(43.5) + 1.6\,(32.0) = 103.4$ ft-kips — *Eq. (9-2)*

$V_u = 1.2\,(12.8) + 1.6\,(9.0) = 29.8$ kips

$N_u = 1.2\,(-2.0) + 1.6\,(-15.2) = -26.7$ kips (tension)

2. Determine shear strength provided by concrete

Since average splitting tensile strength f_{ct} is not specified, $\sqrt{f'_c}$ is reduced by a factor of 0.85 (sand-lightweight concrete) — *11.2.1.2*

$$\phi V_c = \phi 2\left[1 + \frac{N_u}{500A_g}\right]0.85\sqrt{f'_c}\,b_w d$$ — *Eq. (11-8)*

$\phi = 0.75$ — *9.3.2.3*

$$\phi V_c = (0.75)\,2\left[1 + \frac{(-26{,}700)}{500\,(18\times 10.5)}\right]0.85\sqrt{3600}\,(10.5)\,16/1000 = 9.2 \text{ kips}$$

3. Check adequacy of cross-section.

$(V_u - \phi V_c) \leq \phi 8\,(0.85)\sqrt{f'_c}\,b_w d$ — *11.5.7.9*

Note: 0.85 is a factor for lightweight concrete per 11.2.1.2

$(V_u - \phi V_c) = 29.8 - 9.2 = 20.6$ kips

$\phi 8(0.85)\sqrt{f'_c}b_w d = 0.75\times 8\times 0.85\sqrt{3600}\times 10.5\times 16/1000 = 51.4 \text{ kips} > 20.6 \text{ kips}$ O.K.

4. Determine required spacing of U-stirrups

Assuming No. 3 U-stirrups ($A_v = 0.22$ in.^{2}),

$$s\ (\text{req'd}) = \frac{\phi A_v f_{yt} d}{(V_u - \phi V_c)}$$

$$= \frac{0.75 \times 0.22 \times 40 \times 16}{20.6} = 5.1 \text{ in.}$$

5. Determine maximum permissible spacing of stirrups

$V_u - \phi V_c = 20.6$ kips

$\phi 4(0.85)\sqrt{f'_c} b_w d = 25.7$ kips > 20.6 kips — *11.2.1.2*

Therefore, provisions of 11.5.5.1 apply. — *11.5.5.3*

s (max) of vertical stirrups $\leq d/2 = 8$ in. (governs) — *11.5.5.1*
or ≤ 24 in.

s (max) of No. 3 U-stirrups corresponding to minimum reinforcement area requirements:

$$s\ (\max) = \frac{A_v f_{yt}}{0.75(0.85)\sqrt{f'_c} b_w} = \frac{0.22 \times 40{,}000}{0.75 \times 0.85 \times \sqrt{3600} \times 10.5} = 21.9 \text{ in.}$$ — *11.5.6.3*

$$s\ (\max) = \frac{A_v f_{yt}}{50 b_w} = \frac{0.22\ (40{,}000)}{50\ (10.5)} = 16.8 \text{ in.}$$

s (max) = 8 in. (governs)

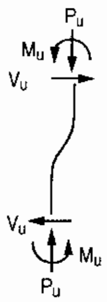
Summary:

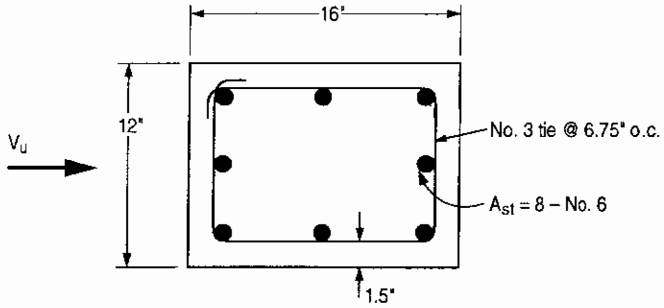
Use No. 3 vertical stirrups @ 5.0 in. spacing.

Example 12.3—Design for Shear - with Axial Compression

A tied compression member has been designed for the given load conditions. However, the original design did not take into account the fact that under a reversal in the direction of lateral load (wind), the axial load, due to the combined effects of gravity and lateral loads, becomes P_u = 10 kips, with essentially no change in the values of M_u and V_u. Check shear reinforcement requirements for the column under (1) original design loads and (2) reduced axial load.

M_u = 86 ft-kips
P_u = 160 kips
V_u = 20 kips
f'_c = 4000 psi
f_{yt} = 40,000 psi





Calculations and Discussion	Code Reference

Condition 1: $P_u = N_u = 160$ kips

1. Determine shear strength provided by concrete

   $d = 16 - [1.5 + 0.375 + (0.750/2)] = 13.75$ in.

   $$\phi V_c = \phi 2\left[1 + \frac{N_u}{2000A_g}\right]\sqrt{f'_c}\, b_w d$$ *Eq. (11-4)*

   $\phi = 0.75$ *9.3.2.3*

   $$\phi V_c = 0.75\,(2)\left[1 + \frac{160{,}000}{2000\,(16\times12)}\right]\sqrt{4000}\,(12)(13.75)/1000 = 22.2 \text{ kips}$$

   $\phi V_c = 22.2 \text{ kips} > V_u = 20 \text{ kips}$

2. Since $V_u = 20 \text{ kips} > \phi V_c/2 = 11.1$ kips, minimum shear reinforcement requirements must be satisfied. *11.5.6.1*

   No. 3 stirrups (A_v = 0.22 in.^{2})

   $$s\,(\max) = \frac{A_v f_{yt}}{0.75\sqrt{f'_c}\,b_w} = \frac{0.22\,(40{,}000)}{0.75\sqrt{4000}\,(12)} = 15.5 \text{ in.}$$ *Eq. (11-13)*

   $$s\,(\max) = \frac{A_v f_{yt}}{50 b_w} = \frac{0.22\,(40{,}000)}{50\,(12)} = 14.7 \text{ in.}$$

   s (max) = d/2 = 13.75/2 = 6.9 in. (governs) *11.5.5.1*

   Therefore, use of s = 6.75 in. is satisfactory.

Example 12.3 (cont'd)	Calculations and Discussion	Code Reference

Condition 2: $P_u = N_u = 10$ kips

1. Determine shear strength provided by concrete.

$$\phi V_c = 0.75\,(2)\left[1 + \frac{(10{,}000)}{2000\,(16\times 12)}\right] \times \sqrt{4000}\,(12)(13.75)/1000 = 16.1 \text{ kips}$$ *Eq. (11-4)*

$\phi V_c = 16.1 \text{ kips} < V_u = 20 \text{ kips}$

Shear reinforcement must be provided to carry excess shear.

2. Determine maximum permissible spacing of No. 3 ties

$$s\,(\max) = \frac{d}{2} = \frac{13.75}{2} = 6.9 \text{ in.}$$ *11.5.4.1*

Maximum spacing, d/2, governs for Conditions 1 and 2.

3. Check total shear strength with No. 3 @ 6.75 in.

$$\phi V_s = \phi A_v f_{yt} \frac{d}{s} = \frac{0.75\,(0.22)\,(40)\,(13.75)}{6.75} = 13.4 \text{ kips}$$ *Eq. (11-15)*

$\phi V_c + \phi V_s = 16.1 + 13.4 = 29.5 \text{ kips} > V_u = 20 \text{ kips}$ O.K.

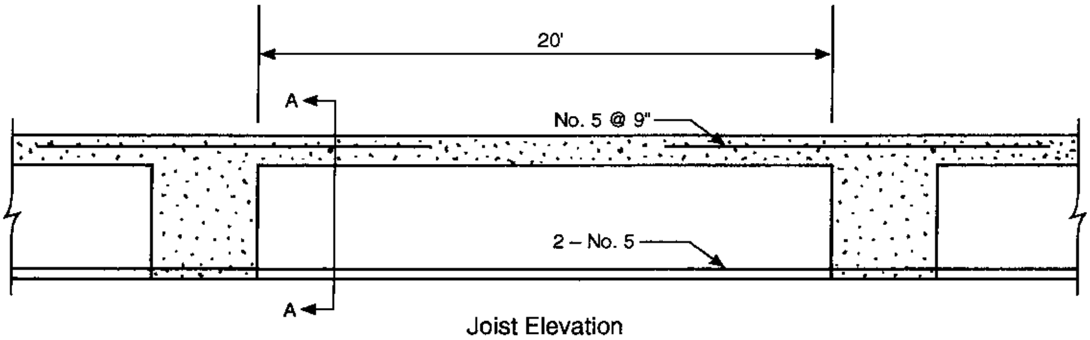
Example 12.4—Design for Shear - Concrete Floor Joist

Check shear requirements in the uniformly loaded floor joist shown below.

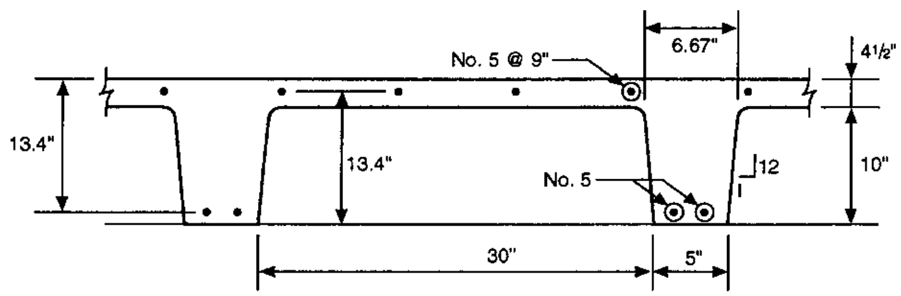
f'_c = 4000 psi
f_{yt} = 40,000 psi
w_d = 77 psf
w_ℓ = 120 psf

Assumed longitudinal reinforcement:

Bottom bars: 2 – No. 5
Top bars: No. 5 @ 9 in.



Joist Elevation



Section

	Calculations and Discussion	Code Reference

1. Determine factored load.

w_u = [1.2 (77) + 1.6 (120)] 35/12 = 830 lb/ft *Eq. (9-2)*

Example 12.4 (cont'd)	Calculations and Discussion	Code Reference

2. Determine factored shear force.

@ distance d from support: *11.1.3.1*

$V_u = 0.83\,(10) - 0.83\,(13.4/12) = 7.4 \text{ kips}$ *8.3.3*

3. Determine shear strength provided by concrete.

According to 8.11.8, V_c may be increased by 10 percent.

Average width of joist web $b_w = (6.67 + 5) / 2 = 5.83$ in.

$\phi V_c = 1.1\phi 2\sqrt{f'_c}\, b_w d$ *8.11.8* *Eq. (11-3)*

$\phi = 0.75$ *9.3.2.3*

$\phi V_c = 1.1\,(0.75)\,2\sqrt{4000}\,(5.83)\,(13.4)/1000 = 8.2 \text{ kips}$

$\phi V_c = 8.2 \text{ kips} > V_u = 7.4 \text{ kips}$ O.K.

Note that minimum shear reinforcement is not required for joist construction defined by 8.11. *11.5.6.1(b)*

Alternatively, calculate V_c using Eq. (11-5)

Compute ρ_w and $V_u d/M_u$ at distance d from support:

$$\rho_w = \frac{A_s}{b_w d} = \frac{(2 \times 0.31)}{(5.83)\,(13.4)} = 0.0079$$

$$M_u \text{ @ face of support} = \frac{w_u \ell_n^2}{11} = \frac{0.83\,(20)^2}{11} = 30.2 \text{ ft-kips}$$ *8.3.3*

$$M_u \text{ @ } d = \frac{w_u \ell_n^2}{11} + \frac{w_u d^2}{2} - \frac{w_u \ell_n d}{2}$$

$$= 30.2 + \frac{(0.83)(13.4/12)^2}{2} - \frac{(0.83)(20)(13.4/12)}{2} = 21.5 \text{ ft-kips}$$

$$\frac{V_u d}{M_u} = \frac{7.4(13.4/12)}{21.5} = 0.38 < 1.0 \quad \text{O.K.}$$ *11.3.2.1*

$$\phi V_c = \phi\, 1.1\left(1.9\sqrt{f'_c} + 2500\,\rho_w \frac{V_u d}{M_u}\right) b_w d \le \phi(1.1)\, 3.5\sqrt{f'_c}\, b_w d$$

$$= 0.75(1.1)\left[1.9\sqrt{4{,}000} + 2500(0.0079)(0.38)\right](5.83)(13.4)/1{,}000$$

$$= 8.2 \text{ kips} < 0.75(1.1)(3.5)\sqrt{4{,}000}(5.83)(13.4)/1{,}000 = 14.3 \text{ kips} \quad \text{O.K.}$$

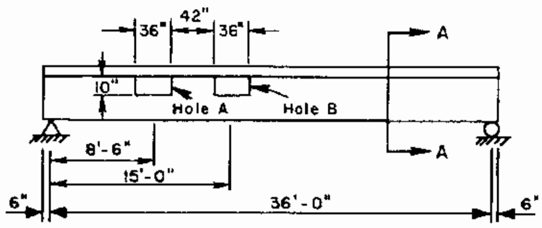
$\phi V_c = 8.2 \text{ kips} > V_u = 7.4 \text{ kips}$ O.K.

Example 12.5—Design for Shear - Shear Strength at Web Openings

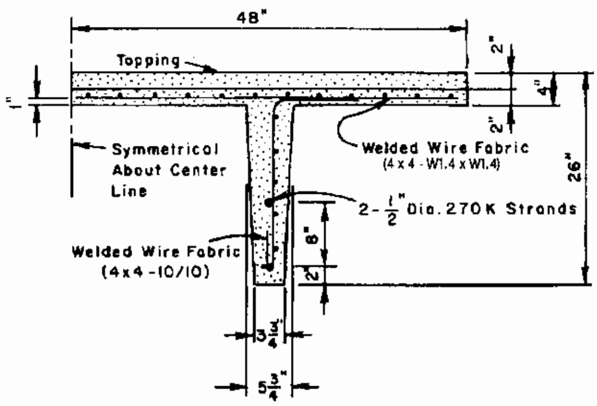
The simply supported prestressed double tee beam shown below has been designed without web openings to carry a factored load w_u = 1520 lb/ft. Two 10-in.-deep by 36-in.-long web openings are required for passage of mechanical and electrical services. Investigate the shear strength of the beam at web opening A.

This design example is based on an experimental and analytical investigation reported in Ref. 12.1.

Beam $f'_c = 6000$ psi
Topping $f'_c = 3000$ psi
$f_{pu} = 270{,}000$ psi
$f_{yt} = 60{,}000$ psi



Elevation



Section A-A
(one-half of the double tee section)

Calculations and Discussion	Code Reference

This example treats only the shear strength considerations for the web opening. Other strength considerations need to be investigated, such as: to avoid slip of the prestressing strand, openings must be located outside the required strand development length, and strength of the struts to resist flexure and axial loads must be checked. The reader is referred to the complete design example in Ref. 12.1 for such calculations. The design example in Ref. 12.1 also illustrates procedures for checking service load stresses and deflections around the openings.

1. Determine factored moment and shear at center of opening A. Since double tee is symmetric about centerline, consider one-half of double tee section.

$$w_u = \frac{1520}{2} = 760 \text{ lb/ft per tee}$$

$$M_u = 0.760\,(36/2)\,(8.5) - 0.760\,(8.5)^2/2 = 88.8 \text{ ft-kips}$$

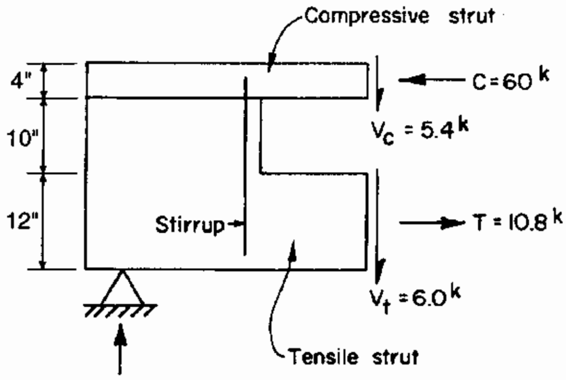
$$V_u = 0.760\,(36/2) - 0.760\,(8.5) = 7.2 \text{ kips}$$

2. Determine required shear reinforcement adjacent to opening. Vertical stirrups must be provided adjacent to both sides of web opening. The stirrups should be proportioned to carry the total shear force at the opening.

$$A_v = \frac{V_u}{\phi f_{yt}} = \frac{7200}{0.75 \times 60{,}000} = 0.16 \text{ in.}^2$$

Use No. 3 U-stirrup, one on each side of opening (A_v = 0.22 in.^{2})

3. Using a simplified analytical procedure developed in Ref. 12.1, the axial and shear forces acting on the "struts" above and below opening A are calculated. Results are shown in the figure below. The reader is referred to the complete design example in Ref. 12.3 for the actual force calculations. Axial forces should be accounted for in the shear design of the struts.



4. Investigate shear strength for tensile strut.

V_u = 6.0 kips

N_u = -10.8 kips

d = 0.8h = 0.8 (12) = 9.6 in. *11.5.3*

b_w = average width of tensile strut = [3.75 + (3.75 + 2 × 12/22)]/2 = 4.3 in.

$$V_c = 2\left(1 + \frac{N_u}{500A_g}\right)\sqrt{f'_c}\, b_w d$$ *Eq. (11-8)*

$$= 2\left(1 - \frac{10{,}800}{500 \times 4.3 \times 12}\right)\sqrt{6000}\,(4.3)\,(9.6)/1000 = 3.72 \text{ kips}$$

$\phi V_c = 0.75\ (3.72) = 2.8$ kips

$V_u = 6.0$ kips $> \phi V_c = 2.8$ kips

Therefore, shear reinforcement is required in tensile strut.

$$A_v = \frac{(V_u - \phi V_c)\ s}{\phi f_{yt} d}$$

$$= \frac{(6.0 - 2.8)\ 9}{0.75 \times 60 \times 9.6} = 0.07 \text{ in.}^2$$

where $s = 0.75h = 0.75 \times 12 = 9$ in. *11.5.5.1*

Use No. 3 single leg stirrups at 9-in. centers in tensile strut, ($A_v = 0.11$ in.^{2}). Anchor stirrups around prestressing strands with 180 degree bend at each end.

5. Investigate shear strength for compressive strut.

$V_u = 5.4$ kips

$N_u = 60$ kips

$d = 0.8h = 0.8\ (4) = 3.2$ in.

$b_w = 48$ in.

$$V_c = 2\left(1 + \frac{N_u}{2000A_g}\right)\sqrt{f'_c}\ b_w d$$ *Eq. (11-4)*

$$= 2\left(1 + \frac{60{,}000}{2000 \times 48 \times 4}\right)\sqrt{3000}\ (48)\ (3.2)/1000 = 19.5 \text{ kips}$$

$\phi V_c = 0.75\ (19.5) = 14.6$ kips

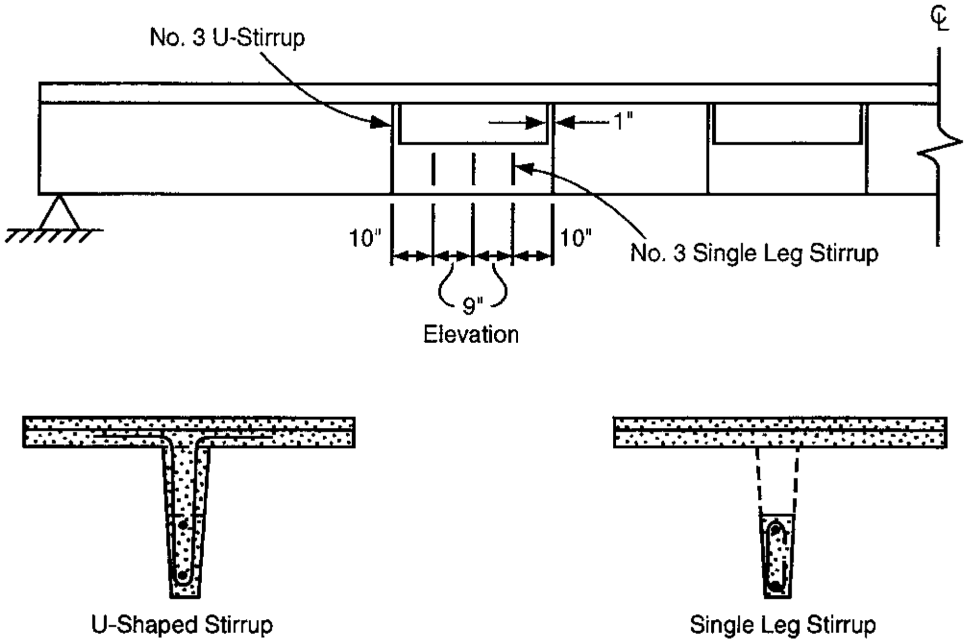
$V_u = 5.4$ kips $< \phi V_c = 14.6$ kips

Therefore, shear reinforcement is not required in compressive strut.

6. Design Summary - See reinforcement details below.

   a. Use U-shaped No. 3 stirrup adjacent to both edges of opening to contain cracking within the struts.

b. Use single-leg No. 3 stirrups at 9-in. centers as additional reinforcement in the tensile strut.

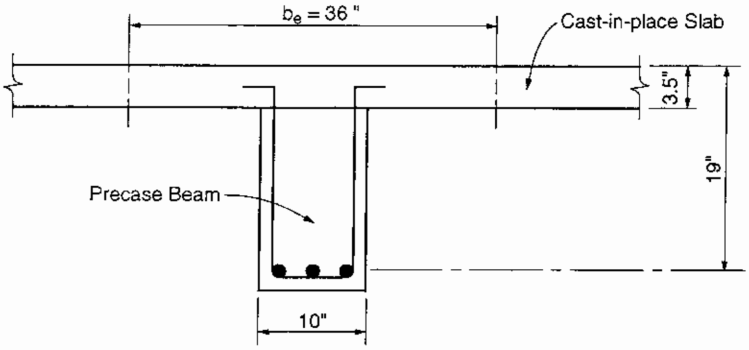


Details of Additional Reinforcement

A similar design procedure is required for opening B.

Example 12.6—Design for Horizontal Shear

For the composite slab and precast beam construction shown, design for transfer of horizontal shear at contact surface of beam and slab for the three cases given below. Assume the beam is simply supported with a span of 30 feet.



f'_c = 3000 psi (normal weight concrete)
f_{yt} = 60,000 psi

Calculations and Discussion	Code Reference

Case I: Service dead load = 315 lb/ft
Service live load = 235 lb/ft
Factored load = 1.2(315) + 1.6 (235) = 754 lb/ft — *Eq. (9-2)*

1. Determine factored shear force V_u at a distance d from face of support:.

V_u = (0.754 × 30/2) – (0.754 × 19/12) = 10.1 kips — *11.1.3.1*

2. Determine horizontal shear strength. — *17.5.3*

$V_u \leq \phi V_{nh}$ — *Eq. (17-1)*

$\phi V_{nh} = \phi\,(80 b_v d)$ — *17.5.3.1 & 17.5.3.2*

$= 0.75\,(80 \times 10 \times 19)/1000 = 11.4$ kips

$V_u = 10.1$ kips $\leq \phi V_{nh} = 11.4$ kips

Therefore, design in accordance with either 17.5.3.1 or 17.5.3.2:

Note: For either condition, top surface of precast beam must be cleaned and free of laitance prior to placing slab concrete.

If top surface of precast beam is intentionally roughened, no ties are required. *17.5.3.1*

If top surface of precast beam is not intentionally roughened, minimum ties are required in accordance with 17.6. *17.5.3.2*

3. Determine required minimum area of ties. *17.6*

$$A_v = \frac{0.75\sqrt{f'_c}\,b_w d}{f_{yt}} \geq \frac{50 b_w s}{f_{yt}}$$ *11.5.5.3*

where s (max) = 4 (3.5) = 14 in. < 24 in. *17.6.1*

$$A_v = \frac{0.75\sqrt{3000}(10)(14)}{60{,}000} = 0.096 \text{ in.}^2 \text{ at 14 in. o.c.}$$

$$\text{Min. } A_v = \frac{50 \times 10 \times 14}{60{,}000} = 0.117 \text{ in.}^2 \text{ at 14 in. o.c.}$$

or 0.10 in.^{2}/ft

Case II: Service dead load = 315 lb/ft
Service live load = 1000 lb/ft
Factored load = 1.2(315) + 1.6 (1000) = 1978 lb/ft *9.2*

1. Determine factored shear force V_u at a distance d from face of support.

$$V_u = (1.98 \times 30/2) - (1.98 \times 19/12) = 26.6 \text{ kips}$$ *11.1.3.1*

2. Determine horizontal shear strength. *17.5.3*

$$V_u = 26.6 \text{ kips} > \phi V_{nh} = \phi\,(80 b_v d) = 11.4 \text{ kips}$$

Therefore, 17.5.2.3 must be satisfied. Minimum ties are required as computed above (A_v = 0.10 in.^{2}/ft).

$$V_{nh} = \phi\,(260 + 0.6\rho_v f_{yt})\,\lambda b_v d$$ *17.5.3.3*

where $\rho_v = \dfrac{A_v}{b_v s} = \dfrac{0.10 \text{ in.}^2}{10 \text{ in. } (12 \text{ in.})}$

$= 0.00083$

$\lambda = 1.0$ (normal weight concrete) *11.7.4.3*

$\phi V_{nh} = 0.75\,(260 + 0.6 \times 0.00083 \times 60{,}000)\,(1.0 \times 10 \times 19)$

$= 0.75\,(290)\,190 = 41.3$ kips

$\phi V_{nh} = 41.3 \text{ kips} < \phi\,(500 b_v d)/1000 = 71.3 \text{ kips}$ O.K. *17.5.3.3*

$V_u = 26.6 \text{ kips} < \phi V_{nh} = 41.3 \text{ kips}$

Therefore, design in accordance with 17.5.2.3:

Contact surface must be intentionally roughened to "a full amplitude of approximately 1/4-in.," and minimum ties provided in accordance with 17.6.

3. Compare tie requirements with required vertical shear reinforcement at distance d from face of support.

$V_u = 26.6$ kips

$V_c = 2\sqrt{f'_c}\, b_w d = 2\sqrt{3000} \times 10 \times 19/1000 = 20.8 \text{ kips}$ *Eq. (11-3)*

$V_u \le \phi\,(V_c + V_s) = \phi V_c + \phi A_v f_{yt} \dfrac{d}{s}$ *Eq. (11-15)*

Solving for A_v/s:

$\dfrac{A_v}{s} = \dfrac{V_u - \phi V_c}{\phi f_{yt} d} = \dfrac{26.6 - (0.75 \times 20.8)}{0.75 \times 60 \times 19} = 0.013 \text{ in.}^2\text{/in.}$

$s_{max} = \dfrac{19}{2} = 9.5 \text{ in.} < 24 \text{ in.}$ *11.5.5.1*

$A_v = 0.013 \times 9.5 = 0.12 \text{ in.}^2$

Provide No. 3 U-stirrups @ 9.5 in. o.c. ($A_v = 0.28$ in.^{2}/ft). This exceeds the minimum ties required for horizontal shear ($A_v = 0.10$ in.^{2}/ft) so the No. 3 U-stirrups @ 9.5 in. o.c. are adequate to satisfy both vertical and horizontal shear reinforcement requirements. Ties must be adequately anchored into the slab by embedment or hooks. See Fig. 12-8.

Case III: Service dead load = 315 lb/ft
Service live load = 3370 lb/ft
Factored load = 1.2(315) + 1.6 (3370) = 5770 lb/ft *9.2*

1. Determine factored shear force V_u at distance d from support.

$$V_u = (5.77 \times 30/2) - (5.77 \times 19/12) = 77.4 \text{ kips}$$ *11.1.3.1*

$$V_u = 77.4 \text{ kips} > \phi(500 b_v d) = 0.75(500 \times 10 \times 19)/1000 = 71.3 \text{ kips}$$ *17.5.3.4*

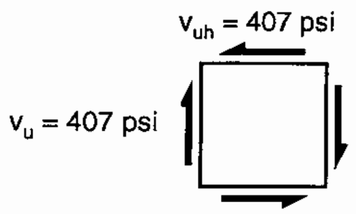
Since V_u exceeds $\phi(500b_vd)$, design for horizontal shear must be in accordance with 11.7.4 - Shear-Friction. Shear along the contact surface between beam and slab is resisted by shear-friction reinforcement across and perpendicular to the contact surface.

As required by 17.5.3.1, a varied tie spacing must be used, based on the actual shape of the horizontal shear distribution. The following method seems reasonable and has been used in the past:

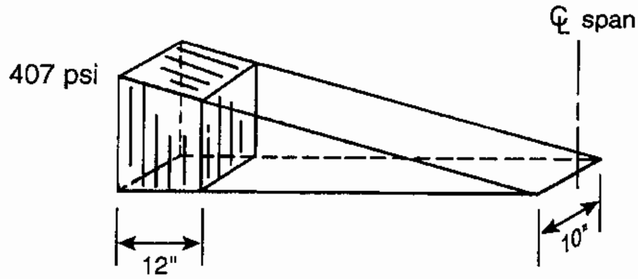
Converting the factored shear force to a unit stress, the factored horizontal shear stress at a distance d from span end is:

$$v_{uh} = \frac{V_u}{b_v d} = \frac{77.4}{10 \times 19} = 0.407 \text{ ksi}$$

The shear "stress block" diagram may be shown as follows:



State of stress at
span end (point A)



Assume that the horizontal shear is uniform per foot of length, then the shear transfer force for the first foot is:

$$V_{uh} = 0.407 \times 10 \times 12 = 48.9 \text{ kips}$$

Example 12.6 (cont'd)	Calculations and Discussion	Code Reference

Required area of shear-friction reinforcement is computed by Eqs. (11-1) and (11-25):

$$V_{uh} \le \phi V_n = \phi A_{vf} f_{yt} \mu$$ *Eq. (11-25)*

$$A_{vf} = \frac{V_{uh}}{\phi f_{yt} \mu}$$

If top surface of precast beam is intentionally roughened to approximately 1/4 in., $\mu = 1.0$. *11.7.4.3*

$$A_{vf} = \frac{48.9}{0.75 \times 60 \times 1.0} = 1.09 \text{ in.}^2/\text{ft}$$

With No. 5 double leg stirrups, $A_{vf} = 0.62 \text{ in.}^2$

$$s = \frac{0.62 \times 12}{1.09} = 6.8 \text{ in.}$$

Use No. 5 U-stirrups @ 6.5 in. o.c. for a minimum distance of d + 12 in. from span end.

If top surface of precast beam is not intentionally roughened, $\mu = 0.6$. *11.7.4.3*

$$A_{vf} = \frac{48.9}{0.75 \times 60 \times 0.6} = 1.81 \text{ in.}^2/\text{ft}$$

$$s = \frac{0.62 \times 12}{1.81} = 4.1 \text{ in.}$$

Use No. 5 U-Stirrups @ 4 in. o.c. for a minimum distance of d + 12 in. from span end.

This method can be used to determine the tie spacing for each successive one-foot length. The shear force will vary at each one-foot increment and the tie spacing can vary accordingly to a maximum of 14 in. toward the center of the span.

Note: Final tie details are governed by vertical shear requirements.

13

Torsion

UPDATE FOR THE '05 CODE

In the 2005 code, the provisions of for torsion design remain essentially unchanged. However, a new section (11.6.7) now permits using alternative procedures for torsion design of solid sections with an aspect ratio, h/b_t, of three or more. Moreover, in addition to standard hooks, 11.6.4.2 allows using seismic hooks to anchor transverse torsional reinforcement.

BACKGROUND

The 1963 code included one sentence concerning torsion detailing. It prescribed use of closed stirrups in edge and spandrel beams and one longitudinal bar in each corner of those closed stirrups. Comprehensive design provisions for torsion were first introduced in the 1971 code. With the exception of a change in format in the 1977 document, the requirements have remained essentially unchanged through the 1992 code. These first generation provisions applied only to reinforced, nonprestressed concrete members. The design procedure for torsion was analogous to that for shear. Torsional strength consisted of a contribution from concrete (T_c) and a contribution from stirrups and longitudinal reinforcement, based on the skew bending theory.

The design provisions for torsion were completely revised in the 1995 code and remain essentially unchanged since then. The new procedure, for solid and hollow members, is based on a thin-walled tube, space truss analogy. This unified approach applies equally to reinforced and prestressed concrete members. Background of the torsion provisions has been summarized by MacGregor and Ghoneim.[13.1] Design aids and design examples for structural concrete members subject to torsion are presented in Ref. 13.2.

For design purposes, the center portion of a solid beam can conservatively be neglected. This assumption is supported by test results reported in Ref. 13.1. Therefore, the beam is idealized as a tube. Torsion is resisted through a constant shear flow q (force per unit length of wall centerline) acting around the centerline of the tube as shown in Fig. 13-1(a). From equilibrium of external torque T and internal stresses:

$$T = 2A_o q = 2A_o \tau t \qquad (1)$$

Rearranging Eq. (1)

$$q = \tau t = \frac{T}{2A_o} \qquad (2)$$

where τ = shear stress, assumed uniform, across wall thickness

t = wall thickness

T = applied torque

A_o = area enclosed within the tube centerline [see Fig. 13-1(b)]

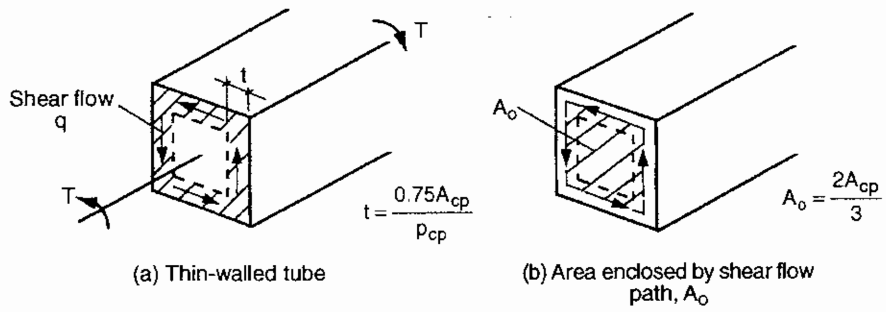


Figure 13-1 Thin-Wall Tube Analogy

When a concrete beam is subjected to a torsional moment causing principal tension larger than $4\sqrt{f'_c}$, diagonal cracks spiral around the beam. After cracking, the tube is idealized as a space truss as shown in Fig. 13-2. In this truss, diagonal members are inclined at an angle θ. Inclination of the diagonals in all tube walls is the same. Note that this angle is not necessarily 45 degree. The resultant of the shear flow in each tube wall induces forces in the truss members. A basic concept for structural concrete design is that concrete is strong in compression, while steel is strong in tension. Therefore, in the truss analogy, truss members that are in tension consist of steel reinforcement or "tension ties." Truss diagonals and other members that are in compression consist of concrete "compression struts." Forces in the truss members can be determined from equilibrium conditions. These forces are used to proportion and detail the reinforcement.

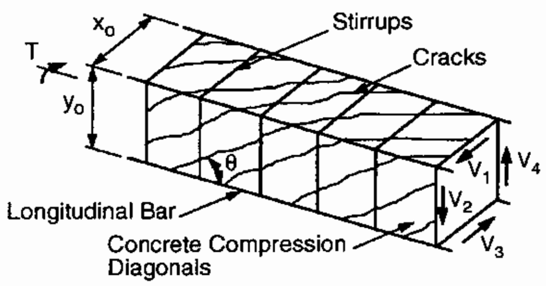


Figure 13-2 Space Truss Analogy

Figure 13-3 depicts a free body extracted from the front vertical wall of the truss of Fig. 13-2. Shear force V_2 is equal to the shear flow q (force per unit length) times the height of the wall y_o. Stirrups are designed to yield when the maximum torque is reached. The number of stirrups intersected is a function of the stirrup spacing s and the horizontal projection $y_o \cot\theta$ of the inclined surface. From vertical equilibrium:

$$V_2 = \frac{A_t f_{yt}}{s} y_o \cot\theta \qquad (3)$$

As the shear flow (force per unit length) is constant over the height of the wall,

$$V_2 = q y_o = \frac{T}{2A_o} y_o \qquad (4)$$

Substituting for V_2 in Eqs. (3) and (4),

$$T = \frac{2A_oA_tf_{yt}}{s}\cot\theta \qquad (5)$$

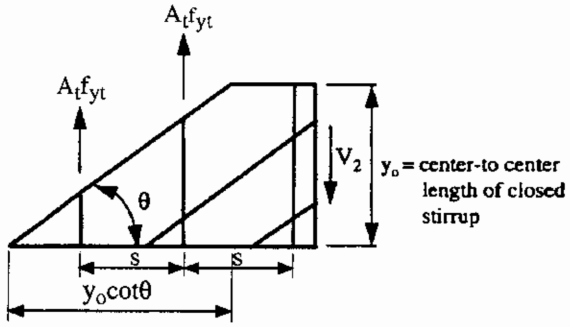


Figure 13-3 Free Body Diagram for Vertical Equilibrium

A free body diagram for horizontal equilibrium is shown in Fig. 13-4. The vertical shear force V_i in wall "i" is equal to the product of the shear flow q times the length of the wall y_i. Vector V_i can be resolved into two components: a diagonal component with an inclination θ equal to the angle of the truss diagonals, and a horizontal component equal to:

$$N_i = V_i\cot\theta$$

Force N_i is centered at the midheight of the wall since q is constant along the side of the element. Top and bottom chords of the free body of Fig. 13-4 are subject to a force $N_i/2$ each. Internally, it is assumed that the longitudinal steel yields when the maximum torque is reached. Summing the internal and external forces in the chords of all the space truss walls results in:

$$\Sigma A_{\ell i}f_y = A_\ell f_y = \Sigma N_i = \Sigma V_i\cot\theta = \Sigma q y_i\cot\theta = \Sigma\frac{T}{2A_o}y_i\cot\theta = \frac{T}{2A_o}\cot\theta\Sigma y_i$$

where $A_\ell f_y$ is the yield force in all longitudinal reinforcement required for torsion.

Rearranging the above equation,

$$T = \frac{2A_oA_\ell f_{y\ell}}{2(x_o + y_o)\cot\theta} \qquad (6)$$

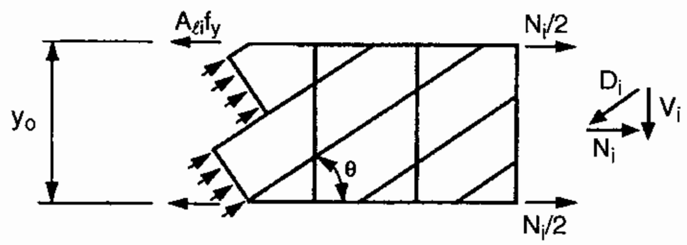


Figure 13-4 Free Body Diagram for Horizontal Equilibrium

11.6.1 Threshold Torsion

Torsion can be neglected if the factored torque T_u is less than $\phi T_{cr}/4$, where T_{cr} is the cracking torque. The cracking torque corresponds to a principal tensile stress of $4\sqrt{f'_c}$. Prior to cracking, thickness of the tube wall "t" and the area enclosed by the wall centerline "A_o" are related to the uncracked section geometry based on the following assumptions:

$$t = \frac{3A_{cp}}{4p_{cp}} \qquad (7)$$

$$A_o = \frac{2A_{cp}}{3} \quad \text{(before cracking)} \qquad (8)$$

where A_{cp} = area enclosed by outside perimeter of concrete cross-section, in.^{2}

p_{cp} = outside perimeter of concrete cross-section, in.

A_o = area within centerline of the thin-wall tube, in.^{2}

Equations (7) and (8) apply to the uncracked section. For spandrel beams and other members cast monolithically with a slab, parts of the slab overhangs contribute to torsional resistance. Size of effective portion of slab to be considered with the beam is illustrated in Fig. R13.2.4.

Substituting for t from Eq. (7), A_o from Eq. (8), and taking $\tau = 4\sqrt{f'_c}$ in Eq. (1), the cracking torque for nonprestressed members can be derived:

$$T_{cr} = 4\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right) \qquad (9)$$

For prestressed concrete members, based on a Mohr's Circle analysis, the principal tensile stress of $4\sqrt{f'_c}$ is reached at $\sqrt{1 + \frac{f_{pc}}{4\sqrt{f'_c}}}$ times the corresponding torque for nonprestressed members. Therefore, the cracking torque for prestressed concrete members is computed as:

$$T_{cr} = 4\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)\sqrt{1 + \frac{f_{pc}}{4\sqrt{f'_c}}} \qquad (10)$$

where f_{pc} = compressive stress in concrete, due to prestress, at centroid of section (also see 2.1)

Similarly, for nonprestressed members subjected to an applied axial force, the principal tensile stress of $4\sqrt{f'_c}$ is reached at $\sqrt{1 + \frac{N_u}{4A_g\sqrt{f'_c}}}$ times the corresponding torque, so that the cracking torque is:

$$T_{cr} = 4\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)\sqrt{1 + \frac{N_u}{4A_g\sqrt{f'_c}}} \qquad (11)$$

where N_u = factored axial force normal to the cross-section (positive for compression)

A_g = gross area of section. For a hollow section, A_g is the area of the concrete only and does not include the area of the void(s) (see 11.6.1).

According to 11.6.1, design for torsion can be neglected if $T_u < \frac{\phi T_{cr}}{4}$, i.e.:

For nonprestressed members:

$$T_u < \phi\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right) \tag{12}$$

For prestressed members:

$$T_u < \phi\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)\sqrt{1+\frac{f_{pc}}{4\sqrt{f'_c}}} \tag{13}$$

For nonprestressed members subjected to an axial tensile or compressive force:

$$T_u < \phi\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)\sqrt{1+\frac{N_u}{4A_g\sqrt{f'_c}}} \tag{14}$$

It is important to note that A_g is to be used in place of A_{cp} in Eqs. (12) through (14) for hollow sections, where for torsion, a hollow section is defined as having one or more longitudinal voids such that $A_g/A_{cp} < 0.95$ (see R11.6.1). The quantity A_g in this case is the area of the concrete only (i.e., the area of the void(s) are not included), based on the outer boundaries prescribed in 13.2.4. The threshold torsion provisions of 11.6.1 were modified in the 2002 code to apply to hollow sections, since results of tests in code Ref. 11.29 indicate that the cracking torque of a hollow section is approximately (A_g/A_{cp}) times the cracking torque of a solid section with the same outside dimensions. Multiplying the cracking torque by (A_g/A_{cp}) a second time reflects the transition from the circular interaction between the inclined cracking loads in shear and torsion for solid members, to the approximately linear interaction for thin-walled hollow sections.

11.6.2 Equilibrium and Compatibility - Factored Torsional Moment T_u

Whether a reinforced concrete member is subject to torsion only, or to flexure combined with shear, the stiffness of that member will decrease after cracking. The reduction in torsional stiffness after cracking is much larger than the reduction in flexural stiffness after cracking. If the torsional moment T_u in a member cannot be reduced by redistribution of internal forces in the structure, that member must be designed for the full torsional moment T_u (11.6.2.1). This is referred to as "equilibrium torsion." See Fig. R11.6.2.1. If redistribution of internal forces can occur, as in indeterminate structures, the design torque can be reduced. This type of torque is referred to as "compatibility torsion." See Fig. R11.6.2.2. Members subject to compatibility torsion need not be designed for a torque larger than the product of the cracking torque times the strength reduction factor ϕ (0.75 for torsion, see 9.3.2.3). For cases of compatibility torsion where $T_u > \phi T_{cr}$ the member can be designed for ϕT_{cr} only, provided redistribution of internal forces is accounted for in the design of the other members of the structure (11.6.2.2). Cracking torque T_{cr} is computed by Eq. (9) for nonprestressed members, by Eq. (10) for prestressed members, and by Eq. (11) for nonprestressed members subjected to an axial tensile or compressive force. For hollow sections, A_{cp} shall not be replaced with A_g in these equations (11.6.2.2).

11.6.2.4-11.6.2.5 Critical Section—In nonprestressed members, the critical section for torsion design is at distance "d" (effective depth) from the face of support. Sections located at a distance less than d from the face of support must be designed for the torque at distance d from the support. Where a cross beam frames into a girder at a distance less than d from the support, a concentrated torque occurs in the girder within distance d. In such cases, the design torque must be taken at the face of support. The same rule applies to prestressed members, except that h/2 replaces distance d, where h is the overall height of member. In composite members, h is the overall height of the composite section.

11.6.3 Torsional Moment Strength

The design torsional strength should be equal to or greater than the required torsional strength:

$$\phi T_n \geq T_u \qquad \text{Eq. (11-20)}$$

The nominal torsional moment strength in terms of stirrup yield strength was derived above [see Eq.(5)]:

$$T_n = \frac{2A_oA_tf_{yt}}{s}\cot\theta \qquad \text{Eq. (11-21)}$$

where $A_o = 0.85A_{oh}$ (this is an assumption for simplicity, see 11.6.3.6)

A_{oh} = area enclosed by centerline of the outermost closed transverse torsional reinforcement as illustrated in Fig. 13-5

θ = angle of compression diagonal, ranges between 30 and 60 degree. It is suggested in 11.6.3.6 to use 45 degree for nonprestressed members and 37.5 degree for prestressed members with prestress force greater than 40 percent of tensile strength of the longitudinal reinforcement.

Note that the definition of A_o used in Eq. (8) was for the uncracked section. Also note that nominal torsional strength T_n is reached after cracking and after the concrete member has undergone considerable twisting rotation. Under these large deformations, part of the concrete cover may have spalled. For this reason, when computing area A_o corresponding to T_n, the concrete cover is ignored. Thus, parameter A_o is related to A_{oh}, the area enclosed by centerline of the outermost closed transverse torsional reinforcement. Area A_o can be determined through rigorous analysis (Ref. 13.3) or simply assumed equal to $0.85A_{oh}$ (see 11.6.3.6).

Substituting for T from Eq. (5) into Eq. (6) and replacing $2(x_o + y_o)$ with p_h (perimeter of centerline of outermost closed transverse torsional reinforcement), the longitudinal reinforcement required to resist torsion is computed as a function of the transverse reinforcement:

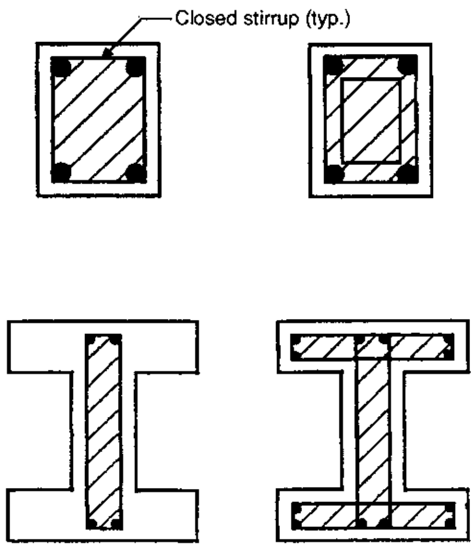


Figure 13-5 Definition of A_{oh}

$$A_\ell = \left(\frac{A_t}{s}\right) p_h \left(\frac{f_{yt}}{f_y}\right) \cot^2\theta \qquad \text{Eq. (11-22)}$$

Note that term (A_t/s) used in Eq. (11-22) is that due to torsion only, and is computed from Eq. (11-21). In members subject to torsion combined with shear, flexure or axial force, the amount of longitudinal and transverse reinforcement required to resist all actions must be determined using the principle of superposition (see 11.6.3.8 and R11.6.3.8). In members subject to flexure, area of longitudinal torsion reinforcement in the flexural compression zone may be reduced to account for the compression due to flexure (11.6.3.9). In prestressed members, the longitudinal reinforcement required for torsion may consist of tendons with a tensile strength $A_{ps}f_{ps}$ equivalent to the yield force of mild reinforcement, $A_\ell f_{y\ell}$, computed by Eq. (11-22).

To reduce unsightly cracking and prevent crushing of the concrete compression struts, 11.6.3.1 prescribes an upper limit for the maximum stress due to shear and torsion, analogous to that due to shear only. In solid sections, stresses due to shear act over the full width of the section, while stresses due to torsion are assumed resisted by a thin-walled tube. See Fig. R11.6.3.1(b). Thus, 11.6.3.1 specifies an elliptical interaction between stresses due to shear and those due to torsion for solid sections as follows:

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right)^2} \leq \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f_c'}\right) \qquad \text{Eq. (11-18)}$$

For hollow sections, the stresses due to shear and torsion are directly additive on one side wall [see Fig. R11.6.3.1(a)]. Thus, the following linear interaction is specified:

$$\left(\frac{V_u}{b_w d}\right) + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right) \leq \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f_c'}\right) \qquad \text{Eq. (11-19)}$$

In Eqs. (11-18) and (11-19), V_c is the contribution of concrete to shear strength of nonprestressed (see 11.3) or prestressed (see 11.4) concrete members. Further, the 2005 code clarifies in 11.5.3 that for prestressed members d should be taken as the distance from extreme compression fiber to centroid of the prestressed and nonprestressed longitudinal tension reinforcement, if any, but need not be taken less than 0.8h.

When applying Eq. (11-19) to a hollow section, if the actual wall thickness t is less than A_{oh}/p_h, the actual wall thickness should be used instead of A_{oh}/p_h (11.6.3.3).

11.6.4 Details of Torsional Reinforcement

Longitudinal and transverse reinforcement are required to resist torsion. Longitudinal reinforcement may consist of mild reinforcement or prestressing tendons. Transverse reinforcement may consist of stirrups, welded wire reinforcement, or spiral reinforcement. To control widths of diagonal cracks, the design yield strength of longitudinal and transverse torsional reinforcement must not exceed 60,000 psi (11.6.3.4).

In the truss analogy illustrated in Fig. 13-2, the diagonal compression strut forces bear against the longitudinal corner reinforcement. In each wall, the component of the diagonal struts, perpendicular to the longitudinal reinforcement is transferred from the longitudinal reinforcement to the transverse reinforcement. It has been observed in torsional tests of beams loaded to destruction that as the maximum torque is reached, the concrete cover spalls.[13.3] The forces in the compression struts outside the stirrups, i.e. within the concrete cover, push out the concrete shell. Based on this observation, 11.6.4.2 specifies that the stirrups should be closed, with 135 degree hooks or seismic hooks as defined in 21.1. Stirrups with 90 degree hooks become ineffective when the concrete cover spalls. Similarly, lapped U-shaped stirrups have been found to be inadequate for resisting torsion due to lack of support when the concrete cover spalls. For hollow sections, the distance from the centerline of the transverse torsional reinforcement to the inside face of the wall of the hollow section must not be less than $0.5A_{oh}/p_h$ (11.6.4.4).

11.6.5 Minimum Torsion Reinforcement

In general, to ensure ductility of nonprestressed and prestressed concrete members, minimum reinforcement is specified for flexure (10.5) and for shear (11.5.6). Similarly, minimum transverse and longitudinal reinforcement is specified in 11.6.5 whenever $T_u > \phi T_{cr}/4$. Usually, a member subject to torsion will also be simultaneously subjected to shear. The minimum area of stirrups for shear and torsion is computed from:

$$(A_v + 2A_t) = 0.75\sqrt{f'_c}\,\frac{b_w s}{f_{yt}} \geq \frac{50 b_w s}{f_{yt}} \qquad \textit{Eq. (11-23)}$$

which now accounts for higher strength concretes (see 11.6.5.2).

The minimum area of longitudinal reinforcement is computed from:

$$A_{\ell,min} = \frac{5\sqrt{f'_c}A_{cp}}{f_y} - \left(\frac{A_t}{s}\right)p_h \frac{f_{yt}}{f_y} \qquad \textit{Eq. (11-24)}$$

but A_t/s (due to torsion only) must not be taken less than $25b_w/f_{yt}$.

11.6.6 Spacing of Torsion Reinforcement

Spacing of stirrups must not exceed the smaller of $p_h/8$ and 12 in. For a square beam subject to torsion, this maximum spacing is analogous to a spacing of about d/2 in a beam subject to shear (11.6.6.1).

The longitudinal reinforcement required for torsion must be distributed around the perimeter of the closed stirrups, at a maximum spacing of 12 in. In the truss analogy, the compression struts push against the longitudinal reinforcement which transfers the transverse forces to the stirrups. Thus, the longitudinal bars should be inside the stirrups. There should be at least one longitudinal bar or tendon in each corner of the stirrups to help transmit the forces from the compression struts to the transverse reinforcement. To avoid buckling of the longitudinal reinforcement due to the transverse component of the compression struts, the longitudinal reinforcement must have a diameter not less than 1/24 of the stirrup spacing, but not less than 3/8 in. (11.6.6.2).

11.6.7 Alternative Design for Torsion

Section 11.6.7 introduced in the 2005 code allows using alternative torsion design procedures for solid sections with h/b_t ratio of three or more. According to 2.1, h is defined as overall thickness of height of members, and b_t is width of that part of cross section containing the closed stirrup resisting torsion. This criterion would be easy to apply to rectangular sections. For other cross sections see discussion below.

An alternative procedure can only be used if its adequacy has been proven by comprehensive tests. Commentary R11.6.7 suggests an alternative procedure, which has been described in detail by Zia and Hsu in Ref 13.4. This procedure is briefly outlined below and its application is also illustrated in Example 13.1.

ZIA-HSU ALTERNATIVE DESIGN PROCEDURE FOR TORSION

Zia-Hsu method for torsion design applies to solid rectangular, box, and flanged sections of prestressed and nonprestressed members. In this procedure L-, T-, inverted T-, and I-shaped sections are subdivided into rectangles, provided that these rectangles include closed stirrups and longitudinal reinforcement required for torsion. Equally important is that the stirrups must overlap adjacent rectangles. This alternative method is most appropriate for precast spandrel beams with a tall stem and a small ledge at the bottom of the stem. In this case, the h/b_t ratio is checked for the vertical stem.

The following steps summarize the procedure:

1. Determine the factored shear force V_u and the factored torsional moment T_u

2. Calculate the shear and torsional constant

$$C_t = \frac{b_w d}{\Sigma x^2 y} \tag{15}$$

where b_w is the web width and d is the distance from extreme compression fiber to centroid of longitudinal prestressed and nonprestressed tension reinforcement, if any, but need not be less than 0.80h for prestressed members. The section has to be divided into rectangular components of dimensions x and y ($x < y$) in such a way that the sum of x^2y terms is maximum. For overhanging flanges, however, the width shall not be taken more than three times the flange thickness (i.e. height).

3. Check the threshold (minimum) torsional moment

$$T_{min} = \phi\, 0.5\sqrt{f_c'}\,\gamma \Sigma x^2 y \tag{16}$$

where $\gamma = \sqrt{1 + \frac{10 f_{pc}}{f_c'}}$ is a prestressing factor and f_{pc} is the average prestressing force in the member after losses.
If $T_u \le T_{min}$, then torsion design is not required. Otherwise proceed to Step 4.

4. Check the maximum permissible torsional moment

$$T_{max} = \frac{\frac{1}{3} C\gamma\sqrt{f_c'}\Sigma x^2 y}{\sqrt{1 + \left(\frac{C\gamma V_u}{30 C_t T_u}\right)^2}} \tag{17}$$

where $C = 12 - 10\frac{f_{pc}}{f_c'}$. If $T_u > T_{max}$, then the section is not adequate and needs to be redesigned.

Options are to use a larger cross section, or increase f_c' or f_{pc}.

5. Calculate nominal torsional moment strength provided by concrete under pure torsion

$$T_c' = 0.8\sqrt{f_c'}\Sigma x^2 y(2.5\gamma - 1.5) \tag{18}$$

6. Calculate the nominal shear strength provided by concrete without torsion $V_c' = 2\sqrt{f_c'} b_w d$ for nonprestressed members and the smaller of V_{ci} and V_{cw} for prestressed members, where V_{ci} and V_{cw} are defined by Eqs (11-10) and (11-12), respectively.

7. Calculate the nominal torsional moment strength provided by concrete under combined loading

$$T_c = \frac{T_c'}{\sqrt{1+\left(\frac{T_c'}{V_c'}\frac{V_u}{T_u}\right)^2}} \qquad (19)$$

8. Calculate the nominal shear strength provided by concrete under combined loading

$$V_c = \frac{V_c'}{\sqrt{1+\left(\frac{V_c'}{T_c'}\frac{T_u}{V_u}\right)^2}} \qquad (20)$$

9. Compute transverse reinforcement for torsion

If $T_u > \phi T_c$, then the area of transverse torsional reinforcement required over distance s equals

$$\frac{A_t}{s} = \frac{T_s}{\alpha_1 x_1 y_1 f_{yt}} \qquad (21)$$

where:
A_t = area of one leg of a closed stirrup resisting torsion

$$T_s = \frac{T_u}{\phi} - T_c$$

$\alpha_t = 0.66 + 0.33\left(\frac{y_1}{x_1}\right)$, but no more than 1.5

x_1 = shorter center-to-center dimension of a closed stirrup
y_1= longer center-to-center dimension of a closed stirrup

10. Compute transverse reinforcement for shear

If $V_u > \phi V_c$, then the area of transverse shear reinforcement required over distance s equals

$$\frac{A_v}{s} = \frac{V_s}{d\, f_{yt}} \qquad (22)$$

where:
A_v = the area of a stirrup (all legs) in section,

$$V_s = \frac{V_u}{\phi} - V_c$$

11. Calculate the total transverse reinforcement

The total transverse reinforcement required for shear and torsion is equal to

$$\frac{A_v}{s} + 2\frac{A_t}{s}$$

but should not be taken less than $\left(\frac{A_v}{s}+2\frac{A_t}{s}\right)_{min}$, which is equal to the smaller of

$50\left(1+12\frac{f_{pc}}{f_c}\right)\frac{b_w}{f_{yt}}$ and $200\frac{b_w}{f_{yt}}$.

12. Calculate longitudinal torsional reinforcement

The area of longitudinal torsional reinforcement required is equal to the larger of

$$A_\ell = 2A_t\left(\frac{x_1+y_1}{s}\right) \qquad (23)$$

and

$$A_\ell = \left[\frac{400xs}{f_y}\left(\frac{T_u}{T_u+\frac{V_u}{3C_t}}\right)-2A_t\right]\left[\frac{x_1+y_1}{s}\right] \qquad (24)$$

However, the value calculated from Eq (24) need not exceed the value obtained when the smaller of

$50\left(1+12\frac{f_{pc}}{f_c}\right)\frac{b_w s}{f_{yt}}$ and $200\frac{b_w s}{f_{yt}}$ is substituted for $2A_t$.

Application of the ACI procedure (11.6) and the Zia-Hsu procedure (Ref. 13.4) is illustrated in Example 13.1

REFERENCES

13.1 MacGregor, J.G., and Ghoneim, M.G., "Design for Torsion," *ACI Structural Journal*, March-April 1995, pp. 211-218.

13.2 Fanella, D. A. and Rabbat, B.G., *Design of Concrete Beams for Torsion*, Engineering Bulletin EB106.02D, Portland Cement Association, Skokie, IL 1997.

13.3 Collins, M.P. and Mitchell, D., *Prestressed Concrete Structures*, Prentice Hall, Englewood Cliffs, NJ, 1991, 766 pp.

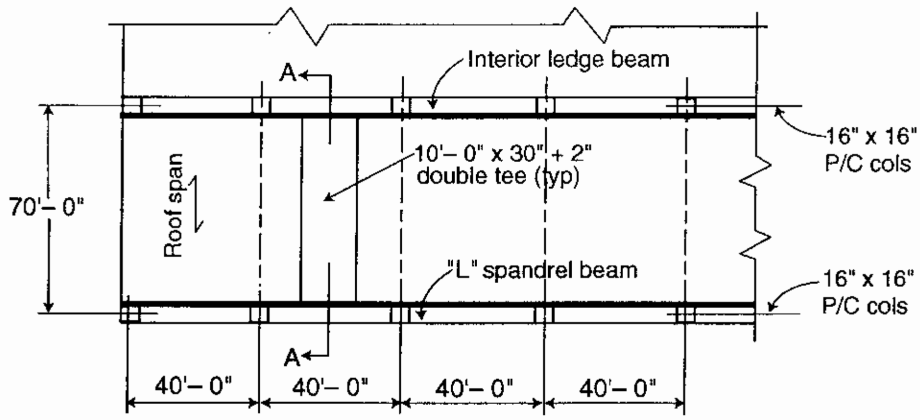
13.4 Zia, P. and Hsu T.T.C., "Design for Torsion and Shear in Prestressed Concrete Flexural Members", *PCI Journal*, Vol. 25, No. 3, May-June 2004, pp. 34-42.

13.5 pcaBeam-Analysis, design, and investigation of reinforced concrete beams and one-way slab systems, Portland Cement Association, Skokie, IL, 2005.

Example 13.1—Precast Spandrel Beam Design for Combined Shear and Torsion

Design a precast, nonprestressed concrete spandrel beam for combined shear and torsion. Roof members are simply supported on spandrel ledge. Spandrel beams are connected to columns to transfer torsion. Continuity between spandrel beams is not provided.

Compare torsional reinforcement requirements using ACI 318-05 provisions, Zia-Hsu alternative design for torsion, and pcaBeam (Ref 13.5) software.



Partial plan of precast roof system

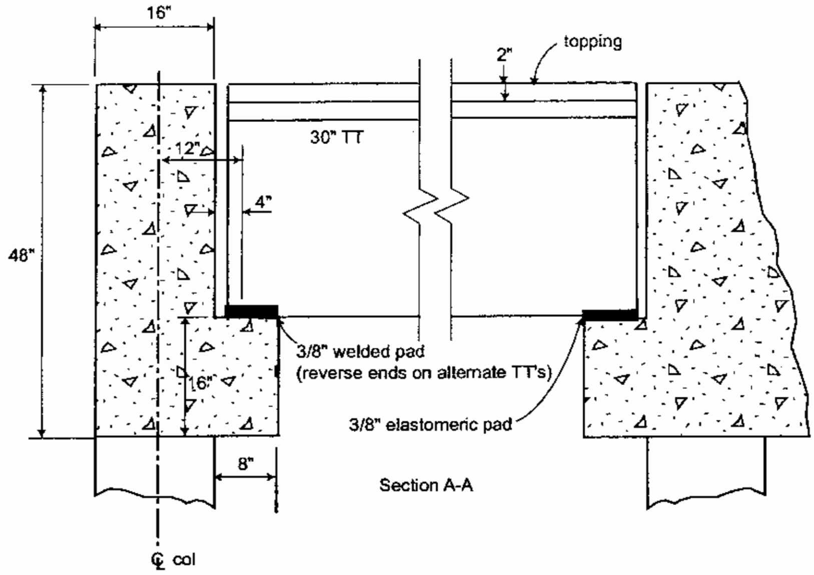
Design Criteria:
Live load = 30 lb/ft^2
Dead load = 90 lb/ft^2 (double tee + topping + insulation + roofing)
f'_c = 5000 psi (w_c = 150 pcf)
f_y = 60,000 psi

Roof members are 10 ft wide double tee units, 30 in. deep with 2 in. topping. Design of these units is not included in this design example. For lateral support, alternate ends of roof members are fixed to supporting beams.

Calculations and Discussion	Code Reference

A. ACI 318 Procedure (11.6)

1. The load from double tee roof members is transferred to the spandrel beam as concentrated forces and torques. For simplicity assume double tee loading on spandrel beam as uniform. Calculate factored loading M_u, V_u, T_u for spandrel beam.



Dead load:

Superimposed	= (0.090) (70)/2	= 3.15
Spandrel	= [(1.33) (4.00) + (1.33) (0.67)] 0.150	= <u>0.94</u>
	Total	= 4.08 kips/ft

Live load = (0.030) (70)/2 = 1.05 kips/ft

Factored load = (1.2) (4.08) + (1.6) (1.05) = 6.58 kips/ft *9.2.1*

At center of span, $M_u = \dfrac{6.58 \times 40^2}{8} = 1316$ ft-kips

End shear V_u = (6.58) (40)/2 = 131.6 kips

Torsional factored load $= 1.2\,(3.15) + 1.2\left(\dfrac{16}{12} \times \dfrac{8}{12} \times 0.150\right) + 1.6\,(1.05) = 5.62$ kips/ft

Eccentricity of double tee reactions relative to centerline of spandrel beam = 8 + 4 = 12 in.

End torsional moment $T_u = 5.62\left(\dfrac{40}{2}\right)\left(\dfrac{12}{12}\right) = 112.4$ ft-kips

Assumed = 45.5 in.

Critical section for torsion is at the face of the support because of concentrated torques applied by the double tee stems at a distance less than d from the face of the support. *11.6.2.4*

Critical section for shear is also at the face of support because the load on the spandrel beam is not applied close to the top of the member and because the concentrated forces transferred by the double tee stems are at a distance less than d from the face of the support. *11.1.3.(b)* *11.1.3.(c)*

Therefore, critical section is 8 in. from column centerline.

At critical section: [20.0 - (8.0/12) = 19.33 ft from midspan]

$V_u = 131.6\,(19.33/20.0) = 127.20$ kips

$T_u = 112.4\,(19.33/20.0) = 108.6$ ft-kips

The spandrel beam must be designed for the full factored torsional moment since it is required to maintain equilibrium. *11.6.2.1*

2. Check if torsion may be neglected *11.6.1*

Torsion may be neglected if $T_u < \dfrac{\phi T_{cr}}{4}$

$\phi = 0.75$ *9.3.2.3*

$$T_{cr} = 4\sqrt{f_c'}\left(\frac{A_{cp}^2}{p_{cp}}\right)$$ *Eq. (9)*

A_{cp} = area enclosed by outside perimeter of spandrel beam, including the ledge
= (16) (48) + (16) (8) = 768 + 128 = 896 in.^{2}

p_{cp} = outside perimeter of spandrel beam
= 2 (16 + 48) + 2 (8) = 144 in.

The limiting value to ignore torsion is:

$$\phi\sqrt{f_c'}\left(\frac{A_{cp}^2}{p_{cp}}\right) = 0.75\sqrt{5000}\left(\frac{896^2}{144}\right)\frac{1}{12{,}000} = 24.6 \text{ ft-kips} < 108.6 \text{ ft-kips}$$ *Eq.(12)*

Torsion must be considered.

3. Determine required area of stirrups for torsion

Design torsional strength must be equal to or greater than the required torsional strength:

$$\phi T_n \geq T_u$$ *Eq. (11-20)*

where

$$T_n = \frac{2A_oA_tf_{yv}}{s}\cot\theta$$ *Eq. (11-21)*

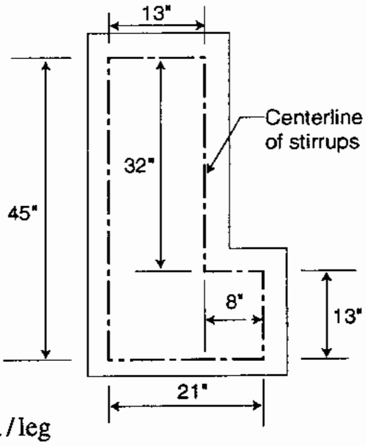
$A_o = 0.85A_{oh}$

A_{oh} = area enclosed by centerline of the outermost closed transverse torsional reinforcement

Assuming 1.25 in. cover (precast concrete exposed to weather) and No. 4 stirrup *7.7.3(a)*

$A_{oh} = (13)(45) + (8)(13) = 689 \text{ in.}^2$

$A_o = 0.85(689) = 585.6 \text{ in.}^2$



For nonprestressed member, use $\theta = 45$ degree *11.6.3.6(a)*

Substituting in Eqs. (11-20) and (11-21)

$$\frac{A_t}{s} = \frac{T_u}{2\phi A_o f_{yv}\cot\theta}$$

$$\frac{A_t}{s} = \frac{(108.6)(12{,}000)}{2(0.75)(586.6)(60{,}000)(1.0)} = 0.025 \text{ in.}^2\text{/in./leg}$$

4. Calculate required area of stirrups for shear

$$V_c = 2\sqrt{f'_c}\,b_wd$$ *Eq. (11-3)*

$$= 2\sqrt{5000}\,(16)(45.5)/1000$$

$$= 102.95 \text{ kips}$$

From Eqs. (11-1) and (11-2)

$$V_s = \frac{V_u}{\phi} - V_c = \frac{127.2}{0.75} - 102.95 = 66.65 \text{ kips}$$

$$\frac{A_v}{s} = \frac{V_s}{f_{yv}d} = \frac{66.65}{60(45.5)} = 0.024 \text{ in.}^2\text{/in.}$$

5. Determine combined shear and torsion stirrup requirements *11.6.3.8*

$$\frac{A_t}{s} + \frac{A_v}{2s} = 0.025 + \frac{0.024}{2} = 0.037 \text{ in.}^2\text{/in./leg}$$

Try No. 4 bar, $A_b = 0.20$ in.^{2}

$$s = \frac{0.20}{0.038} = 5.26 \text{ in.} \quad \text{Use 5 in. minimum spacing.}$$

6. Check maximum stirrup spacing

For torsion spacing must not exceed $p_h/8$ or 12 in.: *11.6.6*

$$p_h = 2\,(13 + 45) + 2\,(6) = 128 \text{ in.}$$

$$\frac{p_h}{8} = \frac{128}{8} = 16 \text{ in.}$$

For shear, spacing must not exceed d/2 or 24 in. $\left(V_s = 66.65 \text{ kips} < 4\sqrt{f'_c}b_wd = 205.9 \text{ kips}\right)$: *11.5.5.1, 11.5.5.3*

$$\frac{d}{2} = \frac{45.5}{2} = 22.75 \text{ in.}$$

Use 5 in. minimum and 12 in. maximum spacing.

7. Check minimum stirrup area

$$(A_v + 2A_t) = 0.75\sqrt{f'_c}\,\frac{b_ws}{f_{yt}} = 0.75\sqrt{5{,}000}\,\frac{(16)(12)}{60{,}000} = 0.17 \text{ in.}^2$$

$$> \frac{50b_ws}{f_{yv}} = \frac{50\,(16)\,(12)}{60{,}000} = 0.16 \text{ in.}^2$$ *Eq. (11-23)*

Area provided $= 2\,(0.20) = 0.40$ in.^{2} > 0.17 in.^{2} O.K.

8. Determine stirrup layout

Since both shear and torsion are zero at the center of span, and are assumed to vary linearly to the maximum value at the critical section, the start of maximum stirrup spacing can be determined by simple proportion.

$$\frac{s\text{ (critical)}}{s\text{ (maximum)}}\,(19.33) = \frac{5}{12}\,(19.33) = 8.05 \text{ ft, say 8 ft from midspan.}$$

9. Check for crushing of the concrete compression struts *11.6.3.1*

$$\sqrt{\left(\frac{V_u}{b_wd}\right)^2 + \left(\frac{T_up_h}{1.7A_{oh}^2}\right)^2} \le \phi\left(\frac{V_c}{b_wd} + 8\sqrt{f'_c}\right)$$ *Eq. (11-18)*

Example 13.1 (cont'd)	Calculations and Discussion	Code Reference

$$\sqrt{\left(\frac{127{,}200}{(16)\,(45.5)}\right)^2 + \left(\frac{(108{,}600 \times 12)\,(128)}{1.7\,(689)^2}\right)^2} = 270.64 \text{ psi} < 10\phi\sqrt{f'_c} = 530 \text{ psi O.K.}$$

10. Calculate longitudinal torsion reinforcement — *11.6.3.7*

$$A_\ell = \left(\frac{A_t}{s}\right) p_h \left(\frac{f_{yt}}{f_y}\right) \cot^2\theta$$ *Eq. (11-22)*

$$A_\ell = (0.025)\,(128)\left(\frac{60}{60}\right)(1.0) = 3.20 \text{ in.}^2$$

Check minimum area of longitudinal reinforcement

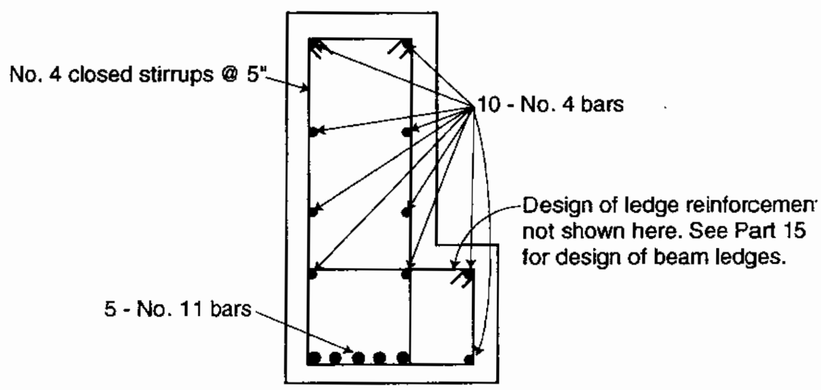
$$A_{\ell,min} = \frac{5\sqrt{f'_c}\,A_{cp}}{f_y} - \left(\frac{A_t}{s}\right) p_h \frac{f_{yt}}{f_y}$$ *Eq. (11-24)*

$$\left(\frac{A_t}{s}\right) \text{ must not be less than } \frac{25b_w}{f_{yt}} = \frac{25\,(16)}{60{,}000} = 0.007 \text{ in.}^2\text{/in.}$$ *11.6.5.3*

$$A_{\ell,min} = \frac{5\sqrt{5000}\,(896)}{60{,}000} - (0.025)\,(122) = 2.23 \text{ in.}^2 < 3.20 \text{ in.}^2$$

The longitudinal reinforcement required for torsion must be distributed around the perimeter of the closed stirrups, at a maximum spacing of 12 in. The longitudinal bars should be inside the stirrups. There should be at least one longitudinal bar in each corner of the stirrups. Select 12 bars. — *11.6.6.2*

$$\text{Area of each longitudinal bar} = \frac{3.17}{12} = 0.264 \text{ in.}^2$$ Use No. 5 bars



11. Size combined longitudinal reinforcement

Use No. 5 bars in sides and top corners of spandrel beam. Note that two of the twelve longitudinal bars (bars at the bottom of the web) required for torsion are to be combined with the ledge reinforcement. Design of the ledge reinforcement is not shown here. See Part 15 of this document for design of beam ledges.

Determine required flexural reinforcement, assuming tension-controlled behavior.

$\phi = 0.90$ 9.3.2

From Part 7,

$$R_n = \frac{M_u}{\phi bd^2} = \frac{1316 \times 12{,}000}{0.9 \times 16 \times 45.5^2} = 529.73 \text{ psi}$$

$$\rho = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right)$$

$$= \frac{0.85 \times 5}{60}\left(1 - \sqrt{1 - \frac{2 \times 529.73}{0.85 \times 5000}}\right) = 0.0095$$

$$A_s = \rho bd = 0.0095 \times 16 \times 45.5 = 6.92 \text{ in.}^2$$

As bottom reinforcement at midspan, provide (2/12) of the longitudinal torsion reinforcement in addition to the flexural reinforcement.

$$\left(\frac{2}{12}\right)(3.20) + 6.92 = 7.45 \text{ in.}^2$$

As bottom reinforcement at end of span, provide (2/12) of the longitudinal torsion reinforcement plus at least (1/3) the positive reinforcement for flexure: 12.11.1

$$\left(\frac{2}{12}\right)(3.20) + \left(\frac{6.92}{3}\right) = 2.84 \text{ in.}^2$$

Use 5-No. 11 bars ($A_s = 7.80 \text{ in.}^2 > 7.45 \text{ in.}^2$)

Check if section is tension-controlled, based on provided reinforcement.

From a strain compatibility analysis, conservatively assuming that the section is subjected to flexure only (see Eq. (8) in Part 8),

$$\varepsilon_t = 0.003\left(\frac{\beta_1}{1-\sqrt{1-\dfrac{40}{17}\dfrac{R_n}{f_c'}}}-1\right) = 0.003\left(\frac{0.80}{1-\sqrt{1-\dfrac{40}{17}\dfrac{529.73}{5000}}}-1\right) = 0.015 > 0.005$$

Therefore, section is tension-controlled, and $\phi = 0.90$. *10.3.4*

Note that for strain compatability analysis including the effects of torsion, see Ref. 13.3.

Extend 2-No. 11 bars to end of girder ($A_s = 3.12$ in.^{2} > 2.84 in.^{2})

Note that the longitudinal torsion reinforcement must be adequately anchored.

B. Zia-Hsu Alternative Torsion Design (Ref. 13-4)

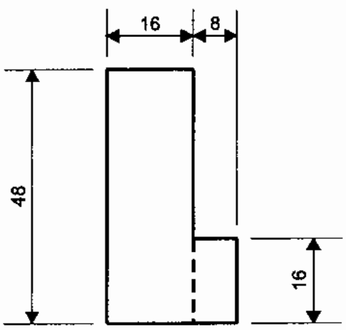
For comparison torsional reinforcement requirements will be determined according to Zia-Hsu alternative design procedure for torsion design. Since a non-prestressed member is considered then $f_{pc} = 0$ will be used.

1. Determine the factored shear force V_u and the factored torsional moment T_u

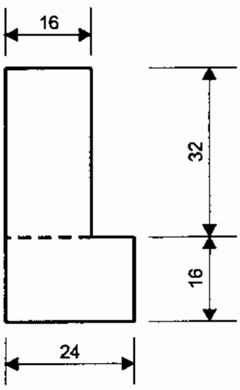
   Based on calculations in A (ACI 318 Procedure):
   $V_u = 127.2$ kips
   $T_u = 108.6$ ft-kips $= 1303.2$ in.-kips

2. Calculate the shear and torsional constant

   Compute the largest $\sum x^2y$ value. Consider Options A and B.



Option A



Option B

For Option A:

$$\sum x^2y = \left(16^2 \times 48\right) + \left(8^2 \times 16\right) = 13{,}312 \text{ in.}^3$$

For Option B:

$$\Sigma x^2 y = \left(16^2 \times 32\right) + \left(16^2 \times 24\right) = 14{,}336 \text{ in.}^3$$

$$C_t = \frac{b_w d}{\Sigma x^2 y} = \frac{16 \times 45.5}{14{,}336} = 0.05078 \frac{1}{\text{in.}}$$ *Eq. (15)*

3. Check the minimum torsional moment

$$T_{min} = \phi 0.5 \sqrt{f'_c}\, \gamma \Sigma x^2 y$$
$$= 0.75 \times 0.5 \times \sqrt{5000} \times 1.0 \times 14{,}336/12{,}000 = 31.76 \text{ft-kips}$$ *Eq. (16)*

where $\gamma = \sqrt{1 + \frac{10 f_{pc}}{f'_c}} = \sqrt{1 + \frac{10 \times 0}{5000}} = 1.0$

Since $T_u > T_{min}$ torsion design is required.

4. Check the maximum torsional moment

$$T_{max} = \frac{\frac{1}{3} C \gamma \sqrt{f'_c} \Sigma x^2 y}{\sqrt{1 + \left(\frac{C \gamma V_u}{30 C_t T_u}\right)^2}}$$
$$= \frac{\frac{1}{3} \times 12.0 \times 1.0 \times \sqrt{5000} \times 14{,}336}{\sqrt{1 + \left(\frac{12.0 \times 1.0 \times 127.2}{30 \times 0.05078 \times 1303.2}\right)^2}} \times \frac{1}{12{,}000} = 267.88 \text{ft-kips}$$ *Eq. (17)*

where $C = 12 - 10 \frac{f_{pc}}{f'_c} = 12 - 10 \frac{0}{f'_c} = 12.0$

This section is adequate for torsion as $T_u < T_{min}$.

5. Calculate nominal torsional moment strength provided by concrete under pure torsion *Eq. (18)*

$$T'_c = 0.8 \sqrt{f'_c} \Sigma x^2 y (2.5\gamma - 1.5)$$
$$= \frac{0.8 \times \sqrt{5000} \times 14{,}336 \times (2.5 \times 1.0 - 1.5)}{12{,}000} = 67.58 \text{ ft-kips}$$

6. Calculate the nominal shear strength provided by concrete without torsion

$$V_c' = 2\sqrt{f_c'}b_wd = 2 \times \sqrt{5000} \times 16 \times 45.5/1000 = 102.95 \text{ kips}$$

7. Calculate the nominal torsional moment strengths under combined loading

$$T_c = \frac{T_c'}{\sqrt{1+\left(\frac{T_c'}{V_c'}\frac{V_u}{T_u}\right)^2}} = \frac{67.58}{\sqrt{1+\left(\frac{67.58}{102.95}\frac{127.2}{108.6}\right)^2}} = 53.58 \text{ ft-kips}$$

Eq. (19)

8. Calculate the nominal shear strengths under combined loading

$$V_c = \frac{V_c'}{\sqrt{1+\left(\frac{V_c'}{T_c'}\frac{T_u}{V_u}\right)^2}} = \frac{102.95}{\sqrt{1+\left(\frac{102.95}{67.58}\frac{108.6}{127.2}\right)^2}} = 62.75 \text{ kips}$$

Eq. (20)

9. Compute transverse reinforcement for torsion

$$T_u = 108.6 \text{ ft-kips} > \phi T_c = 0.75 \times 53.32 = 39.99 \text{ ft-kips.}$$

Area of transverse torsional reinforcement required over distance s equals

$$\frac{A_t}{s} = \frac{T_s}{\alpha_t x_1 y_1 f_{yt}} = \frac{1097.8}{1.50 \times 13 \times 45 \times 60} = 0.0208 \frac{\text{in.}^2/\text{in.}}{\text{leg}}$$

Eq. (21)

where:

$$T_s = \frac{T_u}{\phi} - T_c = \frac{108.6}{0.75} - 53.58 = 91.22 \text{ ft-kips} = 1094.7 \text{ in.-kips}$$

$$\alpha_t = 0.66 + 0.33\left(\frac{y_1}{x_1}\right) = 0.66 + 0.33\left(\frac{45}{13}\right) = 1.80 > 1.50, \text{ use } 1.50$$

$x_1 = 13$ (shorter center-to-center dimension of a closed stirrup),
$y_1 = 45$ (longer center-to-center dimension of a closed stirrup).

10. Compute transverse reinforcement for shear

$V_u = 127.2\text{ kips} > \phi V_c = 0.75 \times 62.75 = 47.06\text{ kips}$

Area of transverse shear reinforcement required over distance s equals *Eq. (22)*

$$\frac{A_v}{s} = \frac{V_s}{d\,f_{yt}} = \frac{106.85}{45.5 \times 60} = 0.0391\frac{\text{in.}^2}{\text{in.}}$$

where:

$$V_s = \frac{V_u}{\phi} - V_c = \frac{127.2}{0.75} - 62.75 = 106.85\text{ ft-kips}$$

11. Calculate the total transverse reinforcement

The total transverse reinforcement required for shear and torsion is equal to

$$\frac{A_v}{s} + 2\frac{A_t}{s} = 0.0391 + 2 \times 0.0208 = 0.0807\frac{\text{in.}^2}{\text{in.}}$$

which is more than the required minimum of

$$\left(\frac{A_v}{s} + 2\frac{A_t}{s}\right)_{min} = 50\left(1 + 12\frac{f_{pc}}{f_c'}\right)\frac{b_w}{f_y} = 50\left(1 + 12\frac{0}{f_c'}\right)\frac{16}{60{,}000} = 0.0133\frac{\text{in.}^2}{\text{in.}}$$

Assuming a two leg stirrup, the area of one leg should be

$$\frac{A_v}{2s} + \frac{A_t}{s} = 0.0391/2 + 0.0208 = 0.0404\frac{\text{in.}^2/\text{in.}}{\text{leg}}$$

12. Calculate longitudinal torsional reinforcement

The area of longitudinal torsional reinforcement required is equal to

$$A_\ell = 2A_t\left(\frac{x_1+y_1}{s}\right) = 2\frac{A_t}{s}(x_1+y_1) = 2 \times 0.0208 \times (13+45) = 2.41\text{ in.}^2$$ *Eq. (23)*

which is greater than the smaller of the following two values

$$A_\ell = \left[\frac{400x}{f_y}\left(\frac{T_u}{T_u + \frac{V_u}{3C_t}}\right) - \frac{2A_t}{s}\right](x_1 + y_1)$$

$$= \left[\frac{400 \times 16}{60{,}000}\left(\frac{1303.2}{1303.2 + \frac{127.2}{3 \times 0.05078}}\right) - 2 \times 0.0208\right](13 + 45) = 1.33 \text{ in.}^2$$

$$A_\ell = \left[\frac{400x}{f_y}\left(\frac{T_u}{T_u + \frac{V_u}{3C_t}}\right) - 50\frac{b_w}{f_y}\right](x_1 + y_1)$$ *Eq. (24)*

$$= \left[\frac{400 \times 16}{60{,}000}\left(\frac{1303.2}{1303.2 + \frac{127.2}{3 \times 0.05078}}\right) - 0.0133\right](13 + 45) = 3.00 \text{ in.}^2$$

C. pcaBeam Solution

Torsional reinforcement requirements obtained from pcaBeam program are presented graphically in Fig. 13-6. The diagram represents combined shear and torsion capacity in terms of required and provided reinforcement area. The upper part of the diagram is related to the transverse reinforcement and shows that at the face of the support the required reinforcement is

$$\frac{A_v}{s}+2\frac{A_t}{s}=0.076\frac{\text{in.}^2}{\text{in.}}$$

The lower part of the diagram is related to the torsional longitudinal reinforcement and shows that $A_\ell = 3.40$ in.^{2} is required for torsional reinforcement at the face of the support. As shown in Fig. (13-6), close to the supports, Eq. (11-22) governs the required amount of longitudinal torsional reinforcement. As expected, as T_u decreases, so does A_ℓ. However, where Eq. (11-24) for $A_{\ell,min}$ starts to govern, the amount of longitudinal reinforcement increases, although T_u decreases toward the midspan. This anomaly occurs where the minimum required transverse reinforcement governs.

Torsional reinforcement requirements are compared in Table 13-1. Transverse reinforcement requirements are in good agreement. Higher differences are observed for longitudinal reinforcement. The small discrepancy between ACI 318-05 and pcaBeam program (also based on ACI) can be attributed to numerical round-off errors and to fixed 1.5 in. side cover assumed in pcaBeam.

Table 13-1 Comparison of required torsional reinforcement

Required reinforcement	ACI 318-05	Zia-Hsu	pcaBeam
$\left(\frac{A_v}{s}+2\frac{A_t}{s}\right)\left[\frac{\text{in.}^2}{\text{in.}}\right]$	0.074	0.081	0.076
$A_\ell\ (\text{in.}^2)$	3.20	2.41	3.40

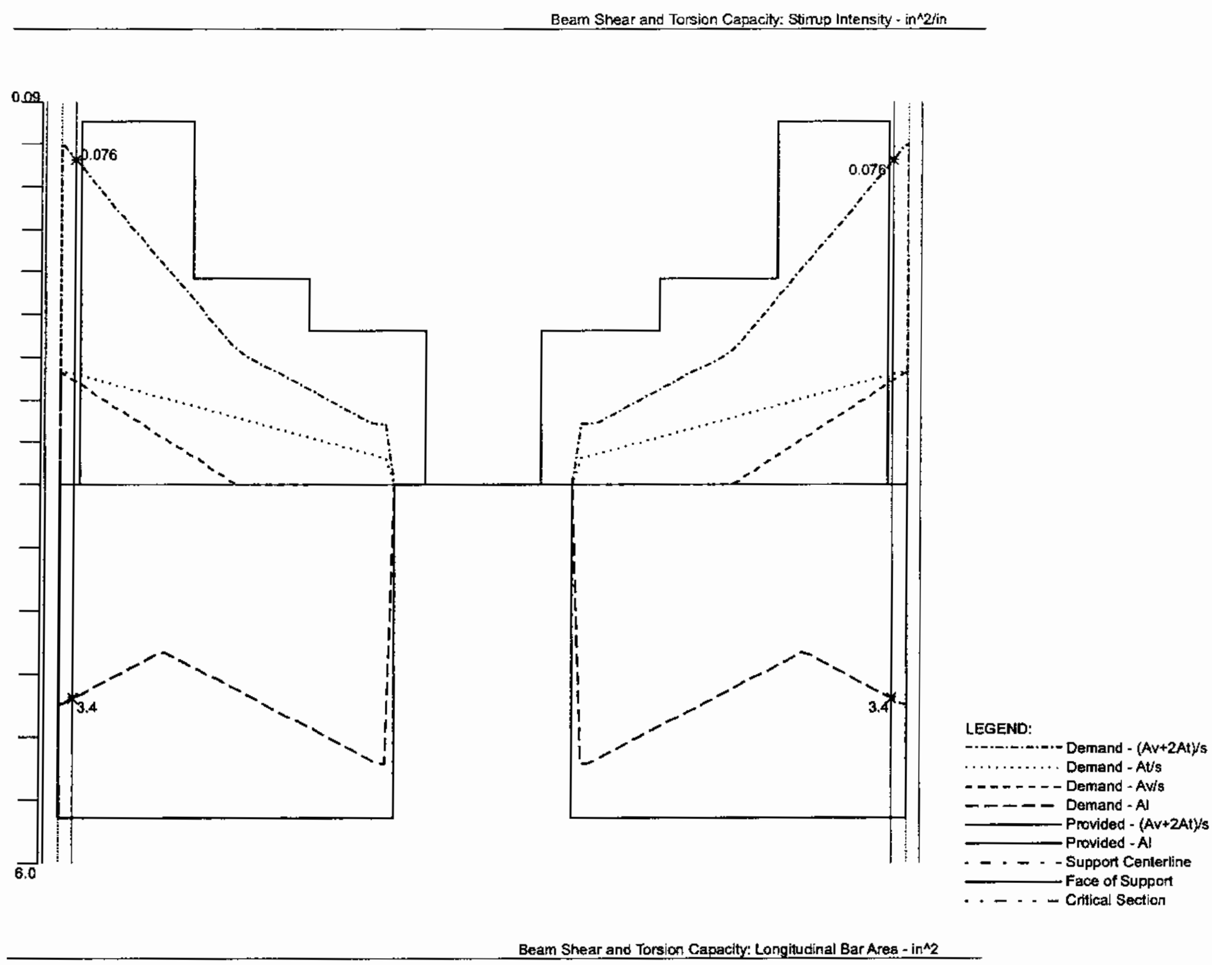


Figure 13-6 Torsional Reinforcement Requirements Obtained from pcaBeam Program

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14

Shear Friction

GENERAL CONSIDERATIONS

Provisions for shear friction were introduced in ACI 318-71. With the publication of ACI 318-83, 11.7 was completely rewritten to expand the shear-friction concept to include applications (1) where the shear-friction reinforcement is placed at an angle other than 90 degrees to the shear plane, (2) where concrete is cast against concrete not intentionally roughened, and (3) with lightweight concrete. In addition, a performance statement was added to allow "any other shear-transfer design methods" substantiated by tests. It is noteworthy that 11.9 refers to 11.7 for the direct shear-transfer in brackets and corbels; see Part 15.

11.7 SHEAR-FRICTION

The shear-friction concept provides a convenient tool for the design of members for direct shear where it is inappropriate to design for diagonal tension, as in precast connections, and in brackets and corbels. The concept is simple to apply and allows the designer to visualize the structural action within the member or joint. The approach is to assume that a crack has formed at an expected location, as illustrated in Fig. 14-1. As slip begins to occur along the crack, the roughness of the crack surface forces the opposing faces of the crack to separate. This separation is resisted by reinforcement (A_{vf}) across the assumed crack. The tensile force ($A_{vf}f_y$) developed in the reinforcement by this strain induces an equal and opposite normal clamping force, which in turn generates a frictional force ($\mu A_{vf}f_y$) parallel to the crack to resist further slip.

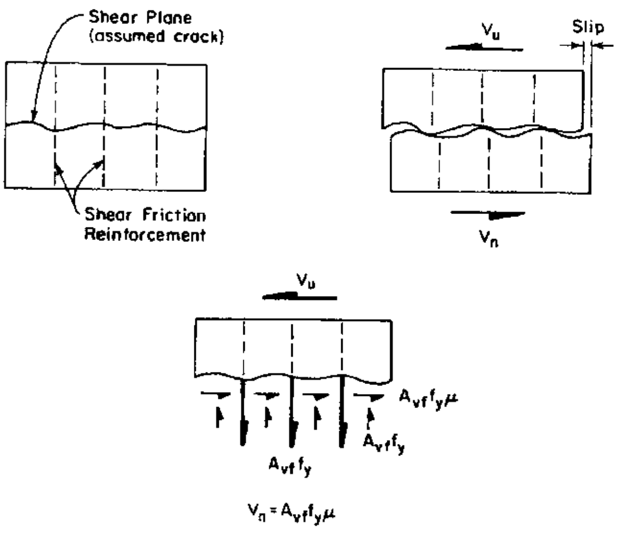


Figure 14-1 Idealization of the Shear-Friction Concept

11.7.1 Applications

Shear-friction design is to be used where direct shear is being transferred across a given plane. Situations where shear-friction design is appropriate include the interface between concretes cast at different times, an interface between concrete and steel, and connections of precast constructions, etc. Example locations of direct shear transfer and potential cracks for application of the shear-friction concept are shown in Fig. 14-2 for several types of members. Successful application of the concept depends on proper selection of location of the assumed slip or crack. In typical end or edge bearing applications, the crack tends to occur at an angle of about 20 degrees to the vertical (see Example 14.2).

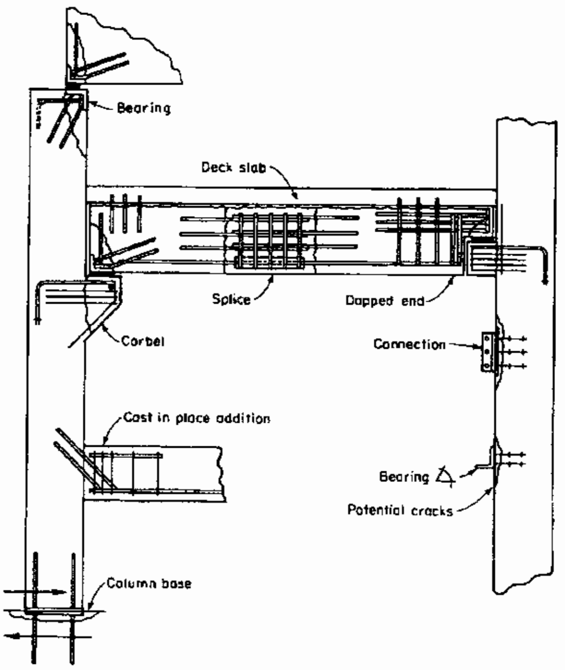


Figure 14-2 Applications of the Shear-Friction Concept and Potential Crack Locations

11.7.3 Shear-Transfer Design Methods

The shear-friction design method presented in 11.7.4 is based on the simplest model of shear-transfer behavior, resulting in a conservative prediction of shear-transfer strength. Other more comprehensive shear-transfer relationships provide closer predictions of shear-transfer strength. The performance statement of 11.7.3 "...any other shear-transfer design methods..." includes the other methods within the scope and intent of 11.7. However, it should be noted that the provisions of 11.7.5 through 11.7.10 apply to whatever shear-transfer method is used. One of the more comprehensive methods is outlined in R11.7.3. Application of the "Modified Shear-Friction Method" is illustrated in Part 15, Example 15.2. The 1992 edition of the code introduced in 17.5.2.3 a modified shear-friction equation. It applies to the interface shear between precast concrete and cast-in-place concrete.

11.7.4 Shear-Friction Design Method

As with the other shear design applications, the code provisions for shear-friction are presented in terms of the nominal shear-transfer strength V_n for direct application in the basic shear strength relation:

Design shear-transfer strength ≥ Required shear-transfer strength

$$\phi V_n \geq V_u \qquad \textit{Eq. (11-1)}$$

Note that ϕ is 0.75 for shear and torsion (9.3.2.3). Furthermore, it is recommended that $\phi = 0.75$ be used for all design calculations involving shear-friction, where shear effects predominate. For example, 11.9.3.1 specifies the use of $\phi = 0.75$ for all design calculations in accordance with 11.9 (brackets and corbels). The nominal shear strength V_n is computed as:

$$V_n = A_{vf} f_y \mu \qquad \textit{Eq. (11-25)}$$

Combining Eqs. (11-1) and (11-25), the required shear-transfer strength for shear-friction reinforcement perpendicular to the shear plane is:

$$V_u \leq \phi A_{vf} f_y \mu$$

The required area of shear-friction reinforcement, A_{vf}, can be computed directly from:

$$A_{vf} = \frac{V_u}{\phi f_y \mu}$$

The condition where shear-friction reinforcement crosses the shear-plane at an angle α other than 90 degrees is illustrated in Fig. 14-3. The tensile force $A_{vf}f_y$ is inclined to the crack and must be resolved into two components: (1) a clamping component $A_{vf}f_y \sin\alpha$ with an associated frictional force $\mu A_{vf}f_y \sin\alpha$, and (2) a component parallel to the crack that directly resists slip equal to $A_{vf}f_y \cos\alpha$. Adding the two components resisting slip, the nominal shear-transfer strength becomes:

$$V_n = \mu A_{vf} f_y \sin\alpha + A_{vf} f_y \cos\alpha$$

$$= A_{vf} f_y (\mu \sin\alpha + \cos\alpha) \qquad \textit{Eq. (11-26)}$$

Substituting this into Eq. (11-1):

$$V_u \leq \phi[\mu A_{vf} f_y \sin\alpha + A_{vf} f_y \cos\alpha]$$

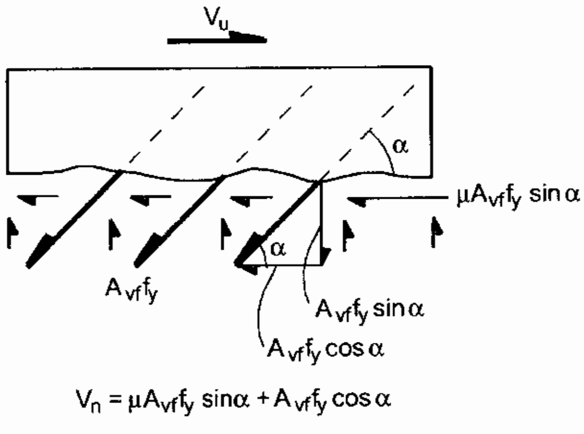


Figure 14-3 Idealization of Inclined Shear-Friction Reinforcement

For shear reinforcement inclined to the crack, the required area of shear-friction reinforcement, A_{vf}, can be computed directly from:

$$A_{vf} = \frac{V_u}{\phi f_y (\mu \sin\alpha + \cos\alpha)}$$

Note that Eq. (11-26) applies only when the shear force V_u produces tension in the shear-friction reinforcement.

The shear-friction method assumes that all shear resistance is provided by friction between crack faces. The actual mechanics of resistance to direct shear are more complex, since dowel action and the apparent cohesive strength of the concrete both contribute to direct shear strength. It is, therefore, necessary to use artificially high values of the coefficient of friction μ in the direct shear-friction equations so that the calculated shear strength will be in reasonable agreement with test results. Use of these high coefficients gives predicted strengths that are a conservative lower bound to test data, as shown in Fig. 14-4. The modified shear-friction design method given in R11.7.3 is one of several more comprehensive methods which provide closer estimates of the shear-transfer strength.

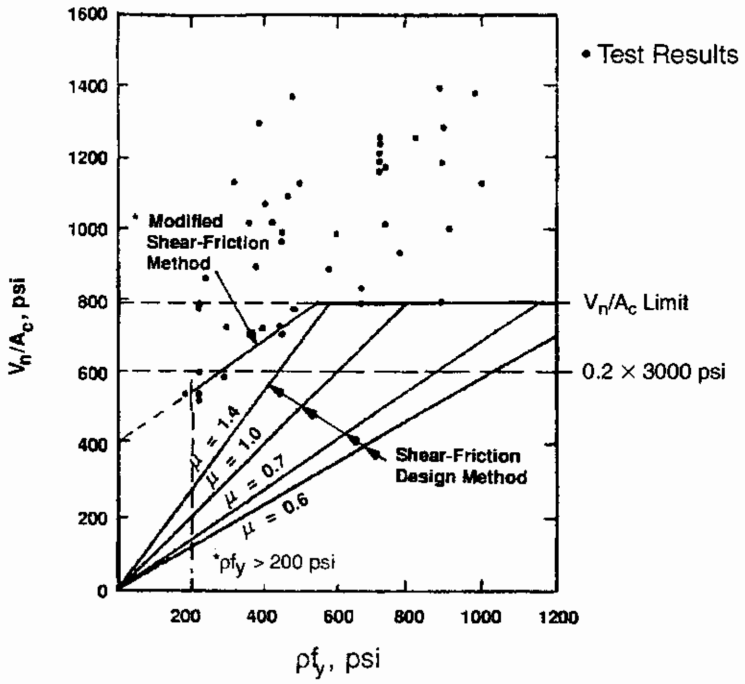


Figure 14-4 Effect of Shear-Friction Reinforcement on Shear Transfer Strength

11.7.4.3 Coefficient of Friction—The "effective" coefficients of friction, μ, for the various interface conditions include a parameter λ which accounts for the somewhat lower shear strength of all-lightweight and sand-lightweight concretes. For example, the μ value for all lightweight concrete (λ = 0.75) placed against hardened concrete not intentionally roughened is 0.6 (0.75) = 0.45. The coefficient of friction for different interface conditions is as follows:

Concrete placed monolithically 1.4λ

Concrete placed against hardened concrete with surface intentionally roughened as specified in 11.7.9 .... 1.0λ

Concrete placed against hardened concrete not intentionally roughened 0.6λ

Concrete anchored to as-rolled structural steel by headed studs or by reinforcing bars (see 11.7.10) 0.7λ

where λ = 1.0 for normal weight concrete, 0.85 for sand-lightweight concrete, and 0.75 for "all light weight" concrete.

11.7.5 Maximum Shear-Transfer Strength

The shear-transfer strength V_n cannot be taken greater than $0.2 f'_c$, nor 800 psi times the area of concrete section resisting shear transfer. This upper limit on V_n effectively limits the maximum reinforcement, as shown in Fig. 14-4. Also, for lightweight concretes, 11.9.3.2.2 limits the shear-transfer strength V_n along the shear plane for design applications with low shear span-to-depth ratios a_v/d , such as brackets and corbels. This further restriction on lightweight concrete is illustrated in Example 14.1.

11.7.7 Normal Forces

Equations (11-25) and (11-26) assume that there are no forces other than shear acting on the shear plane. A certain amount of moment is almost always present in brackets, corbels, and other connections due to eccentricity of loads or applied moments at connections. In case of moments acting on a shear plane, the flexural tension stresses and flexural compression stresses are in equilibrium. There is no change in the resultant compression $A_{vf}f_y$ acting across the shear plane and the shear-transfer strength is not changed. It is therefore not necessary to provide additional reinforcement to resist the flexural tension stresses, unless the required flexural tension reinforcement exceeds the amount of shear-transfer reinforcement provided in the flexural tension zone.

Joints may also carry a significant amount of tension due to restrained shrinkage or thermal shortening of the connected members. Friction of bearing pads, for example, can cause appreciable tensile forces on a corbel supporting a member subject to shortening. Therefore, it is recommended, although not generally required, that the member be designed for a minimum direct tensile force of at least $0.2V_u$ in addition to the shear. This minimum force is required for design of connections such as brackets or corbels (see 11.9.3.4), unless the actual force is accurately known. Reinforcement must be provided for direct tension according to 11.7.7, using $A_s = N_{uc}/\phi f_y$, where N_{uc} is the factored tensile force.

Since direct tension perpendicular to the assumed crack (shear plane) detracts from the shear-transfer strength, it follows that compression will add to the strength. Section 11.7.7 acknowledges this condition by allowing a "permanent net compression" to be added to the shear-friction clamping force, $A_{vf}f_y$. It is recommended, although not required, to use a reduction factor of 0.9 for strength contribution from such compressive loads.

11.7.8 — 11.7.10 Additional Requirements

Section 11.7.8 requires that the shear-friction reinforcement be "appropriately placed" along the shear plane. Where no moment acts on the shear plane, uniform distribution of the bars is proper. Where a moment exists, the reinforcement should be distributed in the flexural tension zone.

Reinforcement should be adequately embedded on both sides of the shear plane to develop the full yield strength of the bars. Since space is limited in thin walls, corbels, and brackets, it is often necessary to use special anchorage details such as welded plates, angles, or cross bars. Reinforcement should be anchored in confined concrete. Confinement may be provided by beam or column ties, "external" concrete, or special added reinforcement.

In 11.7.9, if coefficient of friction μ is taken equal to 1.0λ, concrete at the interface must be roughened to a full amplitude of approximately 1/4 in. This can be accomplished by raking the plastic concrete or by bushhammering or chiseling hardened concrete surfaces.

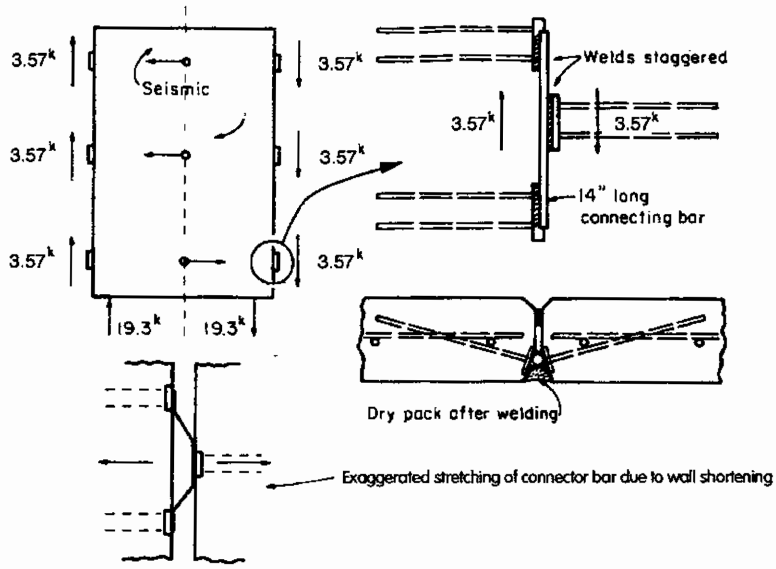
A final requirement of 11.7.10, often overlooked, is that structural steel interfaces must be clean and free of paint. This requirement is based on tests to evaluate the friction coefficient for concrete anchored to unpainted structural steel by studs or reinforcing steel (μ = 0.7). Data are not available for painted surfaces. If painted surfaces are to be used, a lower value of μ would be appropriate.

DESIGN EXAMPLES

In addition to Examples 14.1 and 14.2 of this part, shear-friction design is also illustrated for direct shear-transfer in brackets and corbels (see Part 15), horizontal shear transfer between composite members (see Part 12) and at column/footing connections (see Part 22).

Example 14.1—Shear-Friction Design

A tilt-up wall panel is subject to the factored seismic shear forces shown below. Design the shear anchors assuming lightweight concrete, $w_c = 95$ pcf. $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.



Calculations and Discussion	Code Reference

1. Design anchor steel using shear-friction method.

   Center plate is most heavily loaded. Try 2 in. × 4 in. × 1/4 in. plate.

   $V_u = 3570$ lb

   $V_u \leq \phi V_n$ — *Eq. (11-1)*

   $V_u \leq \phi (A_{vf} f_y \mu)$ — *Eq. (11-25)*

   For unpainted steel in contact with all lightweight concrete (95 pcf):

   $\mu = 0.7\lambda = 0.7 \times 0.75 = 0.525$ — *11.7.4.3*

   $\phi = 0.75$ — *9.3.2.3*

   Solving for $A_{vf} = \dfrac{V_u}{\phi f_y \mu} = \dfrac{3570}{0.75\,(60{,}000)(0.525)} = 0.15 \text{ in.}^2$

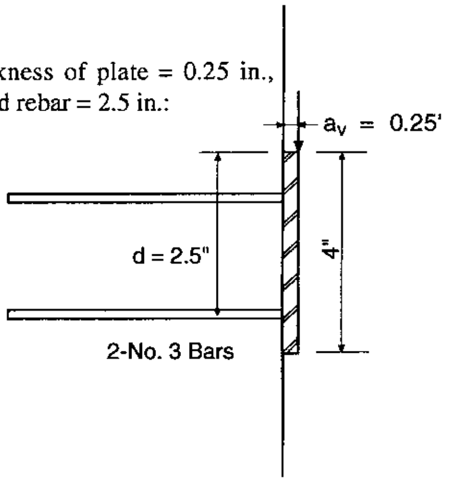
   Use 2-No. 3 bars per plate ($A_{vf} = 0.22 \text{ in.}^2$)

   Note: Weld bars to plates to develop full f_y. Length of bar must be adequate to fully develop bar.

Check maximum shear-transfer strength permitted for connection. For lightweight aggregate concrete: *11.9.3.2.2*

$$V_{n(max)} = \left[0.2 - 0.07\left(\frac{a_v}{d}\right)\right] f'_c b_w d \text{ or } \left[800 - 280\left(\frac{a_v}{d}\right)\right] b_w d$$

For the purposes of the above equations, assume a_v = thickness of plate = 0.25 in., and d = distance from edge of plate to center of farthest attached rebar = 2.5 in.:



$$\frac{a_v}{d} = \frac{0.25}{2.5} = 0.1$$

Assume, for the purposes of the above equations, that $b_w d = A_c$ = contact area of plate:

$b_w d = A_c = 2 \times 4 = 8 \text{ in.}^2$

$V_{n(max)} = [0.2 - 0.07\,(0.1)]\,(4000)\,(8) = 6176 \text{ lb}$

or $V_n = [800 - (280 \times 0.1)]\,(8) = 6176 \text{ lb}$

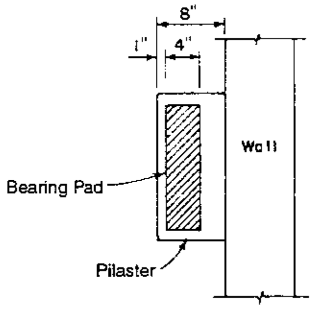
$\phi V_{n(max)} = 0.75\,(6176) = 4632 \text{ lb}$

$V_u = 3570 \text{ lb} \leq \phi V_{n(max)} = 4632 \text{ lb}$ O.K. *Eq. (11-1)*

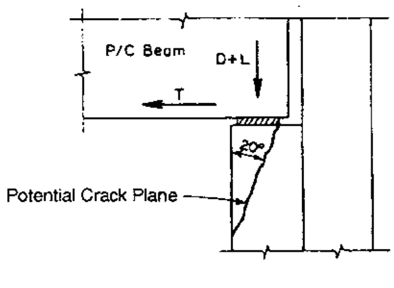
Use 2 in. × 4 in. × $\frac{1}{4}$ in. plates, with 2-No. 3 bars.

Example 14.2—Shear-Friction Design (Inclined Shear Plane)

For the pilaster beam support shown, design for shear transfer across the potential crack plane. Assume a crack at an angle of about 20 degrees to the vertical, as shown below. Beam reactions are D = 25 kips, L = 30 kips. Use T = 20 kips as an estimate of shrinkage and temperature change effects. f'_c = 3500 psi and f_y = 60,000 psi.



Plan



Elevation

Calculations and Discussion	Code Reference

1. Factored loads to be considered:

   Beam reaction R_u = 1.2D + 1.6L = 1.2 (25) + 1.6 (30) = 30 + 48 = 78 kips — *Eq. (9-2)*

   Shrinkage and temperature effects T_u = 1.6 (20) = 32 kips (governs) — *11.9.3.4*
   but not less than 0.2 (R_u) = 0.2 (78) = 15.6 kips

   Note that the live load factor of 1.6 is used with T, due to the low confidence level in determining shrinkage and temperature effects occurring in service. Also, a minimum value of 20 percent of the beam reaction is considered (see 11.9.3.4 for corbel design).

2. Evaluate force conditions along potential crack plane.

   Direct shear transfer force along shear plane:

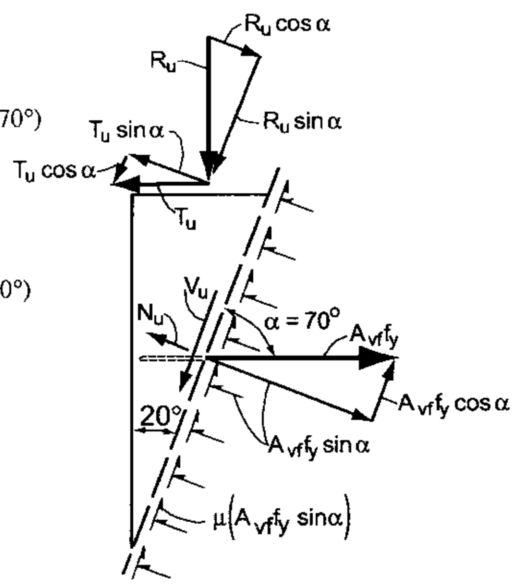
   $V_u = R_u \sin\alpha + T_u \cos\alpha$ = 78 (sin70°) + 32 (cos70°)

   = 73.3 + 11.0 = 84.3 kips

   Net tension (or compression) across shear plane:

   $N_u = T_u \sin\alpha - R_u \cos\alpha$ = 32 (sin70°) - 78 (cos70°)

   = 30.1 - 26.7 = 3.4 kips (net tension)



If the load conditions were such as to result in net compression across the shear plane, it still should not have been used to reduce the required A_{vf}, because of the uncertainty in evaluating the shrinkage and temperature effects. Also, 11.7.7 permits a reduction in A_{vf} only for "permanent" net compression.

3. Shear-friction reinforcement to resist direct shear transfer. Use μ for concrete placed monolithically.

$$A_{vf} = \frac{V_u}{\phi f_y (\mu \sin\alpha + \cos\alpha)}$$ *Eq. (11-26)*

$\mu = 1.4\lambda = 1.4 \times 1.0 = 1.4$ *11.7.4.3*

$$A_{vf} = \frac{84.3}{0.75 \times 60 (1.4 \sin 70° + \cos 70°)} = 1.13 \text{ in.}^2$$ *[μ from 11.7.4.3]*

4. Reinforcement to resist net tension.

$$A_n = \frac{N_u}{\phi f_y (\sin\alpha)} = \frac{3.4}{0.75 \times 60 (\sin 70°)} = 0.08 \text{ in.}^2$$

Since failure is primarily controlled by shear, use $\phi = 0.75$ (see 11.9.3.1 for corbel design).

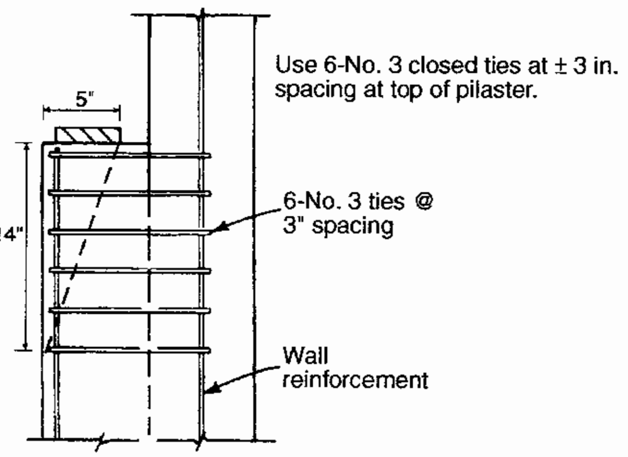
5. Add A_{vf} and A_n for total area of required reinforcement. Distribute reinforcement uniformly along the potential crack plane.

$A_s = 1.13 + 0.08 = 1.21 \text{ in.}^2$

Use No. 3 closed ties (2 legs per tie)

Number required = 1.21 / [2 (0.11)] = 5.5, say 6.0 ties

Ties should be distributed along length of potential crack plane; approximate length = 5/(tan20°) ≈ 14 in.



6. Check reinforcement requirements for dead load only plus shrinkage and temperature effects. Use 0.9 load factor for dead load to maximize net tension across shear plane.

$R_u = 0.9D = 0.9\,(25) = 22.5$ kips, $T_u = 32$ kips

$V_u = 22.5\,(\sin 70°) + 32\,(\cos 70°) = 21.1 + 11.0 = 32.1$ kips

$N_u = 32\,(\sin 70°) - 22.5\,(\cos 70°) = 30.1 - 7.7 = 22.4$ kips (net tension)

$$A_{vf} = \frac{32.1}{0.75 \times 60\,(1.4 \sin 70° + \cos 70°)} = 0.43 \text{ in.}^2$$

$$A_n = \frac{22.4}{0.75 \times 60 \times \sin 70°} = 0.53 \text{ in.}^2$$

$A_s = 0.43 + 0.53 = 0.96 \text{ in.}^2 < 1.21 \text{ in.}^2$

Therefore, original design for full dead load + live load governs.

7. Check maximum shear-transfer strength permitted

$V_{n(max)} = [0.2 f'_c A_c]$ or $[800 A_c]$ *11.7.5*

Taking the width of the pilaster to be 16 in.:

$$A_c = \left(\frac{5}{\sin 20°}\right) \times 16 = 234 \text{ in.}^2$$

$V_{n(max)} = 0.2\,(3500)\,(234)/1000 = 164$ kips (governs)

or $V_{n(max)} = 800\,(234)/1000 = 187$ kips

$\phi V_{n(max)} = 0.75\,(164) = 123$ kips

$V_u = 84.3 \text{ kips} \leq \phi V_{n(max)} = 123$ kips O.K. *Eq. (11-1)*

Blank

15

Brackets, Corbels and Beam Ledges

GENERAL CONSIDERATIONS

Provisions for the design of brackets and corbels were introduced in ACI 318-71. These provisions were derived based on extensive test results. The 1977 edition of the code permitted design of brackets and corbels based on shear friction, but maintained the original design equations. The provisions were completely revised in ACI 318-83, eliminating the empirical equations of the 1971 and 1977 codes, and simplifying design by using the shear-friction method exclusively for nominal shear-transfer strength V_n. From 1971 through 1999 code, the provisions were strictly limited to shear span-to-depth ratio a_v/d less than or equal to 1.0. Since 2002, the code allows the use of the provisions of Appendix A, Strut-and-tie models, to design brackets and corbels with a_v/d ratios less than 2.0, while the provisions of 11.9 continue to apply only for a_v/d ratios less than or equal to 1.0.

11.9 LIMITATIONS OF BRACKET AND CORBEL PROVISIONS

The design procedure for brackets and corbels recognizes the deep beam or simple truss action of these short-shear-span members, as illustrated in Fig. 15-1. Four potential failure modes shown in Fig. 15-1 shall be prevented: (1) Direct shear failure at the interface between bracket or corbel and supporting member; (2) Yielding of the tension tie due to moment and direct tension; (3) Crushing of the internal compression "strut;" and (4) Localized bearing or shear failure under the loaded area.

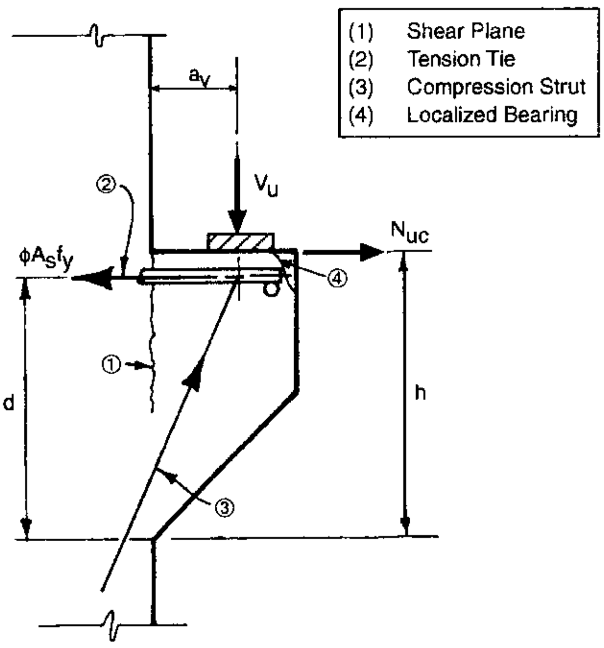


Figure 15-1 Structural Action of Corbel

For brackets and corbels with a shear span-to-depth ratio a_v/d less than 2, the provision of Appendix A may be used for design. The provisions of 11.9.3 and 11.9.4 are permitted with $a_v/d \leq 1$ and the horizontal force $N_{uc} \leq V_n$.

Regardless which design method is used, the provisions of 11.9.2, 11.9.3.2.1, 11.9.3.2.2, 11.9.5, 11.9.6, and 11.9.7 must be satisfied.

When a_v/d is greater than 2.0, brackets and corbels shall be designed as cantilevers subjected to the applicable provisions of flexure and shear.

11.9.1 - 11.9.5 Design Provisions

The critical section for design of brackets and corbels is taken at the face of the support. This section should be designed to resist simultaneously a shear V_u, a moment $M_u = V_u a_v + N_{uc}(h - d)$, and a horizontal tensile force N_{uc} (11.9.3). The value of N_{uc} must be not less than $0.2V_u$, unless special provisions are made to avoid tensile forces (11.9.3.4). This minimum value of N_{uc} is established to account for the uncertain behavior of a slip joint and/or flexible bearings. Also, the tension force N_{uc} typically is due to indeterminate causes such as restrained shrinkage or temperature stresses. In any case it shall be treated as a live load with load factor of 1.6 (11.9.3.4). Since corbel and bracket design is predominantly controlled by shear, 11.9.3.1 specifies that the strength reduction factor ϕ shall be taken equal to 0.75 for all design conditions.

For normal weight concrete, shear strength V_n is limited to the smaller of $0.2f'_c b_w d$ and $800 b_w d$ (11.9.3.2). For lightweight concrete, V_n is limited by the provisions of 11.9.3.2.2, which are somewhat more restrictive than those for normal weight concrete. Tests show that for lightweight concrete, V_n is a function of f'_c and a_v/d.

For brackets and corbels, the required reinforcement is:

A_{vf} = area of shear-friction reinforcement to resist direct shear V_u, computed in accordance with 11.7 (11.9.3.2).

A_f = area of flexural reinforcement to resist moment $M_u = V_u a_v + N_{uc}(h - d)$, computed in accordance with 10.2 and 10.3 (11.9.3.3).

A_n = area of tensile reinforcement to resist direct tensile force N_{uc}, computed in accordance with 11.9.3.4.

Actual reinforcement is to be provided as shown in Fig. 15-2 and includes:

A_{sc} = primary tension reinforcement

A_h = shear reinforcement (closed stirrups or ties)

This reinforcement is provided such that total amount of reinforcement $A_{sc} + A_h$ crossing the face of support is the greater of (a) $A_{vf} + A_n$, and (b) $3A_f/2 + A_n$ to satisfy criteria based on test results.[15.1]

If case (a) controls (i.e., $A_{vf} > 3A_f/2$):

$$A_{sc} = A_{vf} + A_n - A_h$$

$$= A_{vf} + A_n - 0.5(A_{sc} - A_n) \qquad \textit{11.9.4}$$

or $A_{sc} = 2A_{vf}/3 + A_n$ (primary tension reinforcement)

and $A_h = (0.5)(A_{sc} - A_n) = A_{vf}/3$ (closed stirrups or ties) *11.9.4*

If case (b) controls (i.e., $3A_f/2 > A_{vf}$):

$$A_{sc} = 3A_f/2 + A_n - A_h$$

$$= 3A_f/2 + A_n - 0.5\,(A_{sc} - A_n)$$

or $A_{sc} = A_f + A_n$ (primary tension reinforcement)

and $A_h = (0.5)\,(A_{sc} - A_n) = A_f/2$ (closed stirrups or ties)

In both cases (a) and (b), $A_h = (0.5)\,(A_{sc} - A_n)$ determines the amount of shear reinforcement to be provided as closed stirrups parallel to A_{sc} and uniformly distributed within (2/3)d adjacent to A_{sc} per 11.9.4.

A minimum ratio of primary tension reinforcement $\rho_{min} = 0.04 f'_c/f_y$ is required to ensure ductile behavior after cracking under moment and direct tensile force (11.9.5).

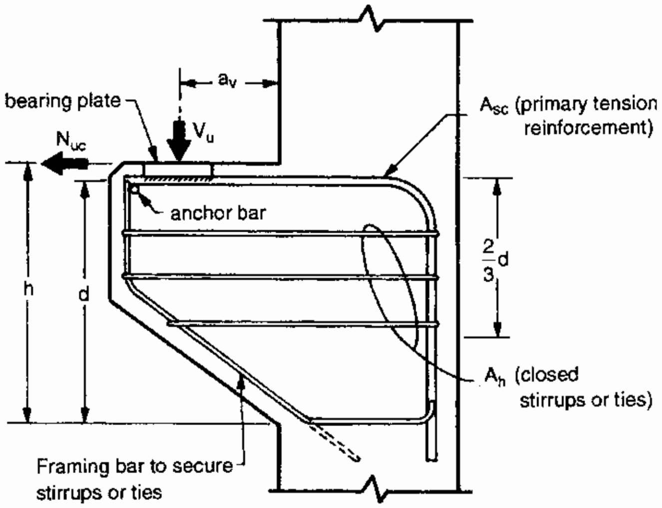


Figure 15-2 Corbel Reinforcement

BEAM LEDGES

Beam with ledges shall be designed for the overall member effects of flexure, shear, axial forces, and torsion, as well as for local effects in the vicinity of the ledge (Refs. 15.2-15.6). The design of beam ledges is not specifically addressed by the code. This section addresses only local failure modes and reinforcement requirements to prevent such failure.

Design of beam ledges is somewhat similar to that of a bracket or corbel with respect to loading conditions. Additional design considerations and reinforcement details need to be considered in beam ledges. Accordingly, even though not specifically addressed by the code, special design of beam ledges is included in this Part. Some failure modes discussed above for brackets and corbels are also shown for beam ledges in Fig. 15-3. However, with beam ledges, two additional failure modes shall be considered (see Fig. 15-3): (5) separation between ledge and beam web near the top of the ledge in the vicinity of the ledge load and (6) punching shear. The vertical load applied to the ledge is resisted by a compression strut. In turn, the vertical component of the inclined compression strut must be picked up by the web stirrups (stirrup legs A_v adjacent to the side face of the web) acting as "hanger" reinforcement to carry the ledge load to the top of beam. At the reentrant corner of the ledge to web intersection, a diagonal crack would extend to the stirrup and run downward next to the stirrup. Accordingly, a slightly larger shear span, a_f, is used to compute the moment due to V_u. Therefore, the critical section for moment is taken at center of beam stirrups, not at face of beam. Also, for beam ledges, the internal moment arm should not be taken greater than 0.8h for flexural strength.

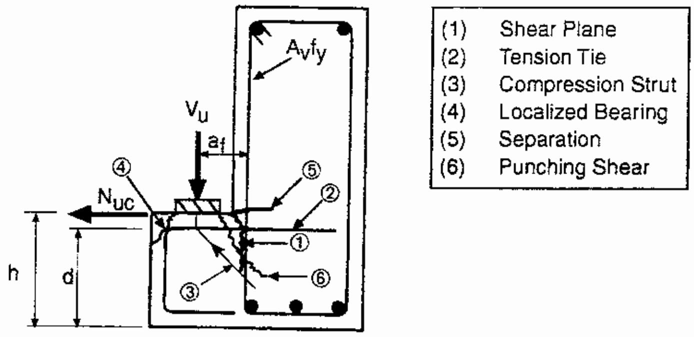


Figure 15-3 Structural Action of Beam Ledge

The design procedure described in this section is based on investigations performed by Mirza and Furlong (Refs. 15.3 to 15.5). The key information needed by the designer is establishing the effective width of ledge for each of the potential failure modes. These effective widths were determined by Mirza and Furlong through analytical investigations, with results verified by large scale testing. Design of beam ledges can also be performed by the strut-and-tie procedure (refer to Part 17 for discussion).

Design to prevent local failure modes requires consideration of the following actions:

1. Shear V_u

2. Horizontal tensile force N_{uc} greater or equal to $0.2V_u$, but not greater than V_u.

3. Moment $M_u = V_u a_f + N_{uc}$ (h-d)

Reinforcement for the different failure modes is determined based on the effective widths or critical sections discussed below. In all cases, the required strengths (V_u, M_u, or N_u) should be less than or equal to the design strengths (ϕV_n, ϕM_n, or ϕN_n). The strength reduction factor ϕ is taken equal to 0.75 for all actions, as for brackets and corbels. The strength requirements for different failure modes are shown below for normal weight concrete. When lightweight aggregate concrete is used, modifications should be made per 11.2.

a. Shear Friction

Parameters affecting determination of the shear friction reinforcement are illustrated in Figure 15-4.

$$V_u \le 0.2\phi f'_c\,(W + 4a_v)d \qquad (1)$$

$$\le \phi\,\mu A_{vf} f_y$$

where

d = effective depth of ledge from centroid of top layer of ledge transverse reinforcement to the bottom of the ledge (see Fig. 15-4)

μ = coefficient of friction per 11.7.4.3.

Note that per 11.7.5, $0.2\,f'_c \le 800$ psi.

If $(W + 4a_v) > S$, then $V_u \le 0.2\phi f'_c\,Sd$, where S is the distance between center of adjacent bearings on the same ledge.

At ledge ends, $V_u \le 0.2\phi f'_c\,(2c)d$, where c is the distance from center of end bearing to the end of the ledge. However, 2c should be less than or equal to the smaller of $(W + 4a_v)$ and S.

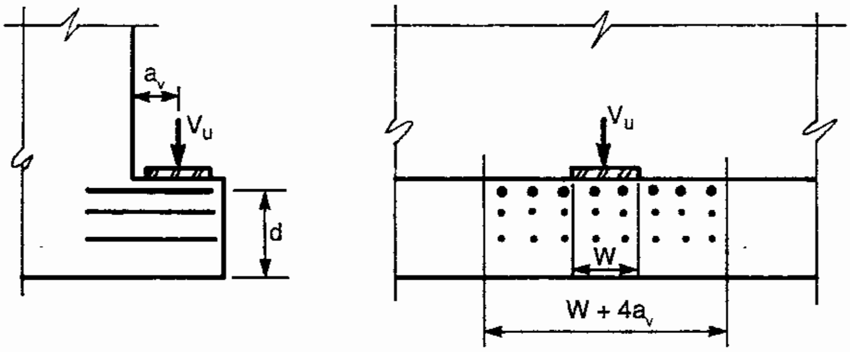


Figure 15-4 Shear Friction

b. Flexure

Conditions for flexure and direct tension are shown in Figure 15-5.

$$V_u a_f + N_{uc}(h\text{-}d) \leq \phi A_f f_y (jd) \tag{2}$$

$$N_{uc} \leq \phi A_n f_y$$

The primary tension reinforcement A_{sc} should equal the greater of $(A_f + A_n)$ or $(2A_{vf}/3 + A_n)$. If $(W + 5a_f) > S$, reinforcement should be placed over distance S. At ledge ends, reinforcement should be placed over distance (2c), where c is the distance from the center of the end bearing to the end of the ledge, but not more than 1/2 $(W + 5a_f)$. Reference 15.5 recommends taking jd = 0.8d.

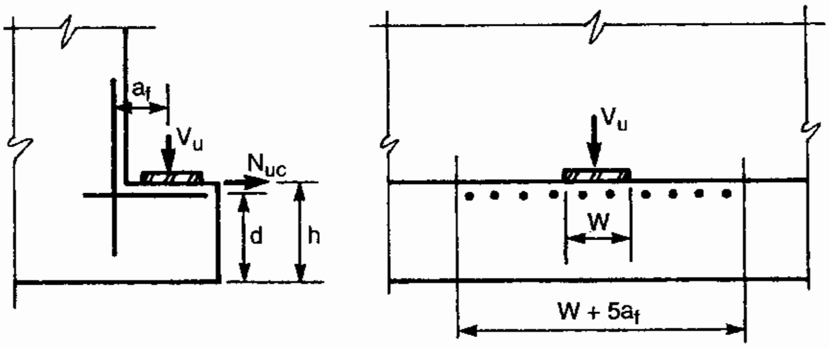


Figure 15-5 Flexure and Direct Tension

c. Punching Shear

Critical perimeter for punching shear design is illustrated in Fig. 15-6.

$$V_u \leq 4\phi\sqrt{f'_c}\,(W + 2L + 2d_f)\,d_f \tag{3}$$

where d_f = effective depth of ledge from top of ledge to center of bottom transverse reinforcement (see Fig. 15-6)

Truncated pyramids from adjacent bearings should not overlap. At ledge ends,

$$V_u \leq 4\phi\sqrt{f'_c}\,(W + L + d_f)\,d_f$$

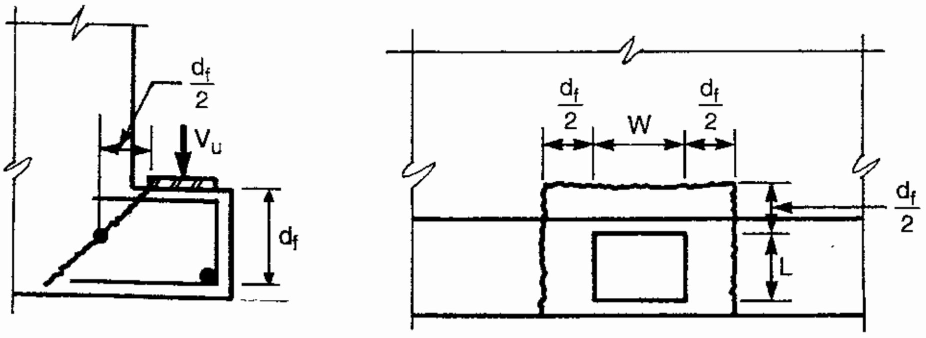


Figure 15-6 Punching Shear

d. Hanger Reinforcement

Hanger reinforcement should be proportioned to satisfy strength. Furthermore, serviceability criteria should be considered when the ledge is subjected to a large number of live load repetitions, as in parking garages and bridges. As shown in Figure 15-7, strength is governed by

$$V_u \leq \phi \frac{A_v f_y}{s} S \qquad (4)$$

where A_V = area of one leg of hanger reinforcement

S = distance between ledge loads

s = spacing of hanger reinforcement

Serviceability is governed by

$$V \leq \frac{A_v\,(0.5 f_y)}{s}\,(W + 3av) \qquad (5)$$

where V is the reaction due to service dead load and live load.

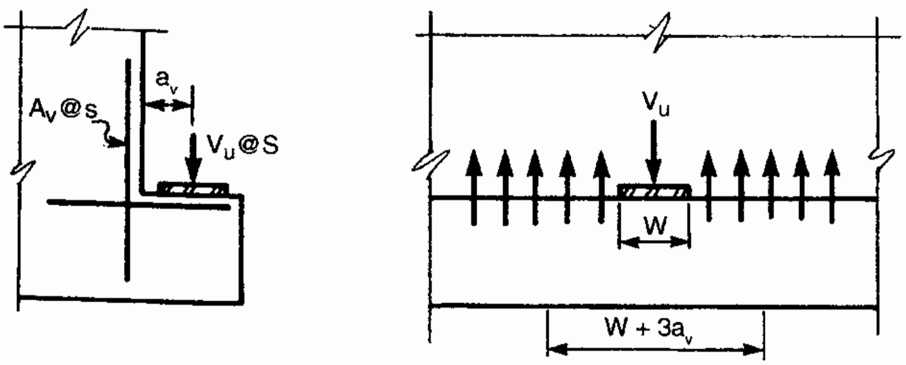


Figure 15-7 Hanger Reinforcement to Prevent Separation of Ledge from Stem

In addition, hanger reinforcement in inverted tees is governed by consideration of the shear failure mode depicted in Figure 15-8:

$$2V_u \leq 2\left[2\phi\sqrt{f'_c}b_f d'_f\right] + \phi \frac{A_v f_y}{s}(W + 2d'_f) \qquad (6)$$

where d'_f = flange depth from top of ledge to center of bottom longitudinal reinforcement (see Fig. 15-8)

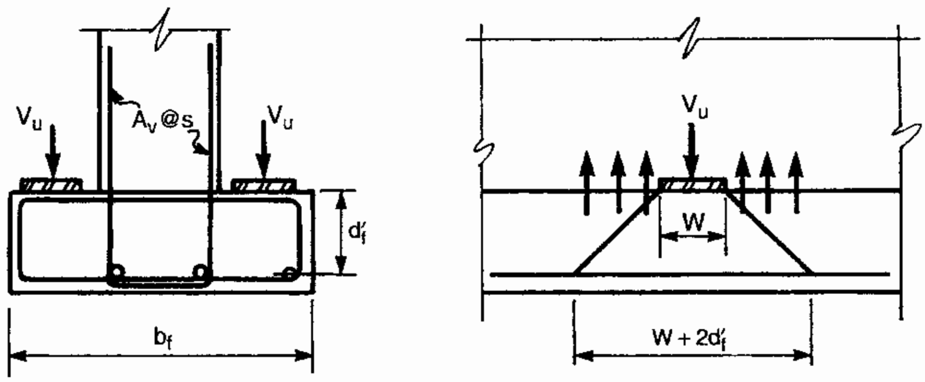


Figure 15-8 Hanger Reinforcement to Prevent Partial Separation of Ledge from Stem and Shear of the Ledge

11.9.6 Development and Anchorage of Reinforcement

All reinforcement must be fully developed on both sides of the critical section. Anchorage within the support is usually accomplished by embedment or hooks. Within the bracket or corbel, the distance between load and support face is usually too short, so that special anchorage must be provided at the outer ends of both primary reinforcement A_{sc} and shear reinforcement A_h. Anchorage of A_{sc} is normally provided by welding an anchor bar of equal size across the ends of A_{sc} (Fig. 15-9(a)) or welding to an armor angle. In the former case, the anchor bar must be located beyond the edge of the loaded area. Where anchorage is provided by a hook or a loop in A_{sc}, the load must not project beyond the straight portion of the hook or loop (Fig. 15-9(b)). In beam ledges, anchorage may be provided by a hook or loop, with the same limitation on the load location (Fig. 15-10). Where a corbel or beam ledge is designed to resist specific horizontal forces, the bearing plate should be welded to A_{sc}.

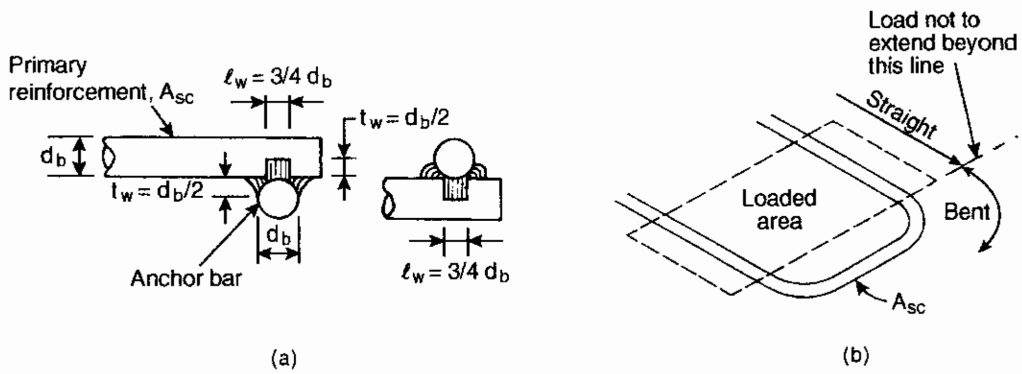


Figure 15-9 Anchorage Details Using (a) Cross-Bar Weld and (b) Loop Bar Detail

The closed stirrups or ties used for A_h must be similarly anchored, usually by engaging a "framing bar" of the same diameter as the closed stirrups or ties (see Fig. 15-2).

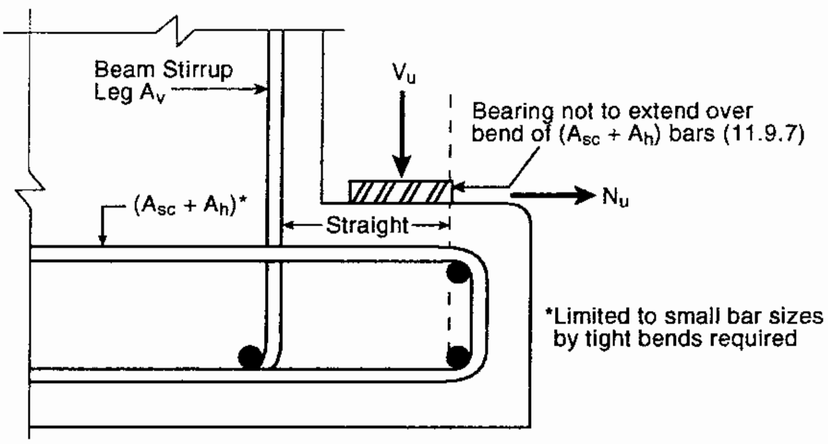


Figure 15-10 Bar Details for Beam Ledge

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15.3 Mirza, Sher Ali, and Furlong, Richard W., "Strength Criteria for Concrete Inverted T-Girder," *Journal of Structural Engineering*, V. 109, No. 8, Aug. 1983, pp. 1836-1853.

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15.6 "Design of Concrete Beams for Torsion," Portland Cement Association, Skokie, Illinois, 1999.

Example 15.1—Corbel Design

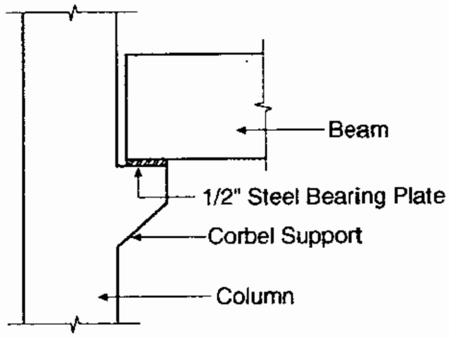
Design a corbel with minimum dimensions to support a beam as shown below. The corbel is to project from a 14-in. square column. Restrained creep and shrinkage create a horizontal force of 20 kips at the welded bearing.

f'_c = 5000 psi (normal weight)

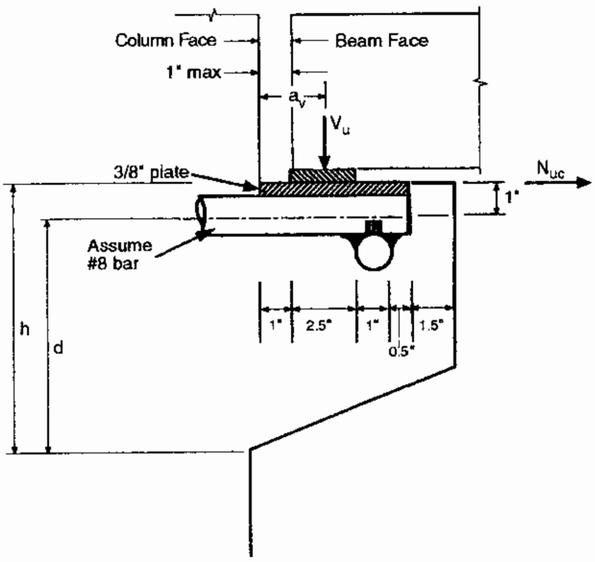
f_y = 60,000 psi

Beam reactions:

DL = 24 kips
LL = 37.5 kips
T = 20 kips



Calculations and Discussion	Code Reference
1. Size bearing plate based on bearing strength on concrete according to 10.17. Width of bearing plate = 14 in.	
$V_u = 1.2\,(24) + 1.6\,(37.5) = 88.8$ kips	*Eq. (9-2)*
$V_u \le \phi P_{nb} = \phi(0.85 f'_c A_1)$	*10.17.1*
$\phi = 0.65$	*9.3.2.4*
$88.8 = 0.65\,(0.85 \times 5 \times A_1) = 2.763A_1$	
$A_1 = \dfrac{88.8}{2.763} = 32.14 \text{ in.}^2$	
Bearing length $= \dfrac{32.14}{14} = 2.30$ in.	
Use 2.5 in. × 14 in. bearing plate.	



2. Determine shear span 'a_v' with 1 in. max. clearance at beam end. Beam reaction is assumed at third point of bearing plate to simulate rotation of supported girder and triangular distribution of stress under bearing pad.

$$a_v = \frac{2}{3}(2.5) + 1.0 = 2.67 \text{ in.}$$

Use a_v = 3 in. maximum.

Detail cross bar just outside outer bearing edge.

Example 15.1 (cont'd)	Calculations and Discussion	Code Reference

3. Determine total depth of corbel based on limiting shear-transfer strength V_n.

V_n is the least of $V_n = 800b_wd$ (governs) — *11.9.3.2.1*

or $V_n = 0.2f'_cb_wd = (0.2 \times 5000)b_wd = 1000b_wd$

Thus, $V_u \le \phi V_n = \phi(800b_wd)$

$$\text{Required } d = \frac{88{,}800}{0.75\,(800 \times 14)} = 10.57 \text{ in.}$$

Assuming No. 8 bar, 3/8 in. steel plate, plus tolerance,

$h = 10.57 + 1.0 = 11.57$ in. Use $h = 12$ in.

For design, $d = 12.0 - 1.0 = 11.0$ in.

$\frac{a_v}{d} = 0.27 < 1$ O.K. — *11.9.1*

Also, $N_{uc} = 1.6 \times 20 = 32.0$ kips (treat as live load)

$N_{uc} < V_u = 88.8$ kips O.K.

4. Determine shear-friction reinforcement A_{vf}. — *11.9.3.2*

$$A_{vf} = \frac{V_u}{\phi f_y \mu} = \frac{88.8}{0.75\,(60)\,(1.4 \times 1)} = 1.41 \text{ in.}^2$$ *11.7.4.1*

5. Determine direct tension reinforcement A_n.

$$A_n = \frac{N_{uc}}{\phi f_y} = \frac{32.0}{0.75 \times 60} = 0.71 \text{ in.}^2$$ *11.9.3.1*

6. Determine flexural reinforcement A_f. — *11.7.4.3*

$M_u = V_ua_v + N_{uc}(h - d) = 88.8\,(3) + 32\,(12 - 11) = 298.4$ in.-kips — *11.9.3.3*

Find A_f using conventional flexural design methods or conservatively use $j_ud = 0.9d$.

$$A_f = \frac{298.4}{0.75\,(60)\,(0.9 \times 11)} = 0.67 \text{ in.}^2$$

Note that for all design calculations, $\phi = 0.75$ — *11.9.3.1*

7. Determine primary tension reinforcement A_s. *11.9.3.5*

$$\frac{2}{3}A_{vf} = \frac{2}{3}(1.41) = 0.94 \text{ in.}^2 > A_f = 0.67 \text{ in.}^2\text{; Therefore, } \frac{2}{3}A_{vf} \text{ controls design}$$

$$A_{sc} = \frac{2}{3}A_{vf} + A_n = 0.94 + 0.71 = 1.65 \text{ in.}^2$$

Use 2-No. 9 bars, $A_{sc} = 2.0 \text{ in.}^2$

Check minimum reinforcement: *11.9.5*

$$\rho_{min} = 0.04\left(\frac{f'_c}{f_y}\right) = 0.04\left(\frac{5}{60}\right) = 0.0033$$

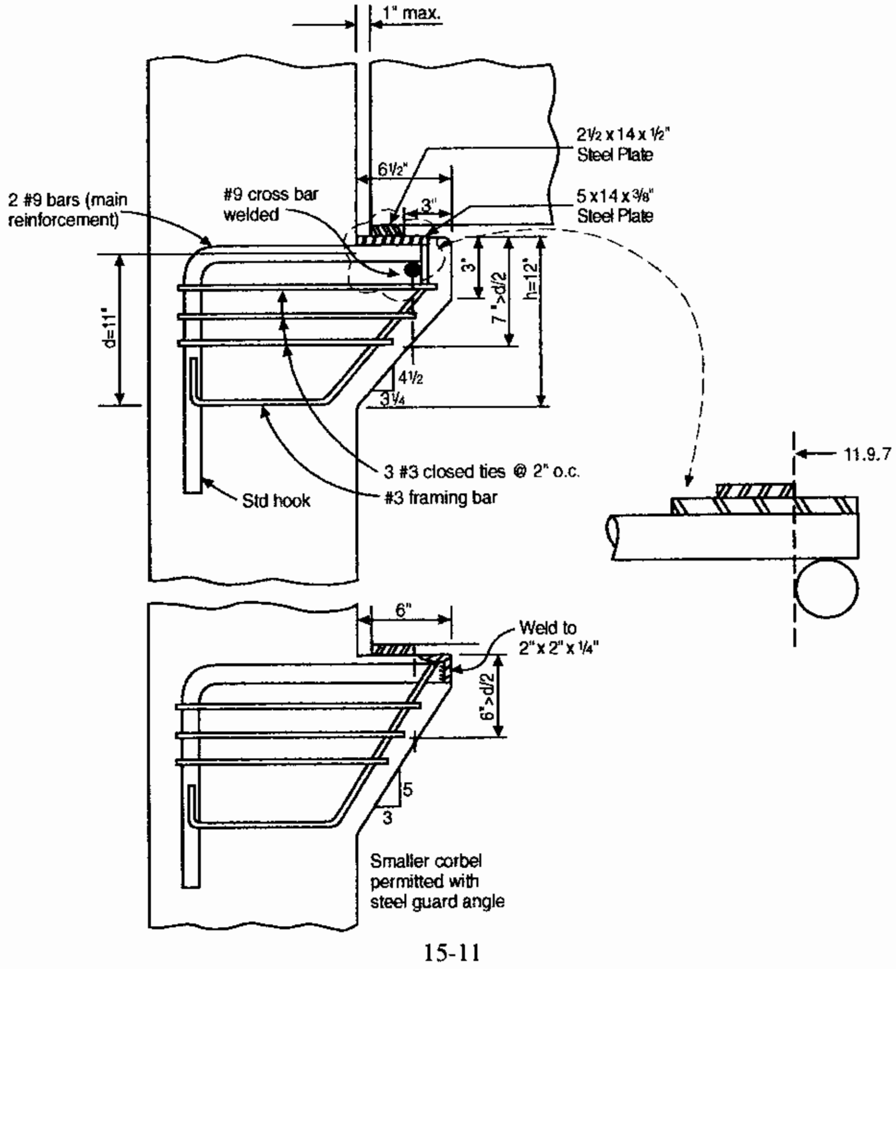
$A_{sc(min)} = 0.0033\,(14)\,(11) = 0.51 \text{ in.}^2 < A_{sc} = 2.0 \text{ in.}^2$ O. K.

8. Determine shear reinforcement A_h *11.9.4*

$A_h = 0.5\,(A_{sc} - A_n) = 0.5\,(2.0 - 0.71) = 0.65 \text{ in.}^2$

Use 3-No. 3 stirrups, $A_h = 0.66 \text{ in.}^2$

Distribute stirrups in two-thirds of effective corbel depth adjacent to A_{sc}.



Example 15.2—Corbel Design . . . Using Lightweight Concrete and Modified Shear-Friction Method

Design a corbel to project from a 14-in.-square column to support the following beam reactions:

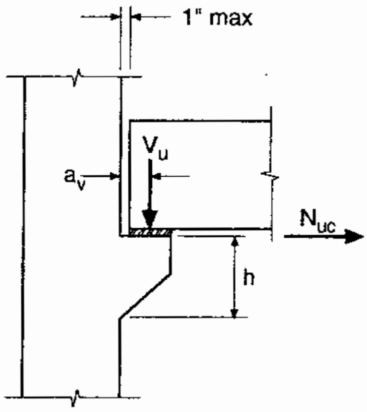
Dead load = 32 kips

Live load = 30 kips

Horizontal force = 24 kips

f'_c = 4000 psi (all lightweight)

f_y = 60,000 psi



Calculations and Discussion	Code Reference

1. Size bearing plate

$V_u = 1.2\,(32) + 1.6\,(30) = 86.4$ kips — *Eq. (9-2)*

$V_u \leq \phi P_{nb} = \phi\,(0.85 f'_c A_1)$ — *10.17.1*

$\phi = 0.65$ — *9.3.2.4*

$86.4 = 0.65\,(0.85 \times 4 \times A_1)$

Solving, $A_1 = 39.1$ in.^{2}

Length of bearing required $= \dfrac{39.1}{14} = 2.8$ in.

Use 14 in. × 3 in. bearing plate.

2. Determine a_v.

Assume beam reaction to act at outer third point of bearing plate, and 1 in. gap between back edge of bearing plate and column face. Therefore:

$a_v = 1 + \dfrac{2}{3}(3) = 3$ in.

3. Determine total depth of corbel based on limiting shear-transfer strength V_n. For easier placement of reinforcement and concrete, try h = 15 in. Assuming No. 8 bar:

$d = 15 - 0.5 - 0.375 = 14.13$ in., say 14 in.

$\dfrac{a_v}{d} = \dfrac{3}{14} = 0.21 < 1.0$ — *11.9.1*

$N_{uc} = 1.6 \times 24 = 38.4$ kips $< V_u = 86.4$ kips O.K.

For lightweight concrete and f'_c = 4000 psi, V_n is the least of: *11.9.3.2.2*

$$V_n = \left(800 - 280\frac{a_v}{d}\right) b_w d = [800 - (280 \times 0.21)]\ 14 \times \frac{14}{1000} = 145.3 \text{ kips}$$

$$V_n = \left(0.2 - 0.07\frac{a_v}{d}\right) f'_c b_w d = [0.2 - 0.07\,(0.21)]\,(4{,}000)\,(14)\,\frac{14}{1000} = 145.3 \text{ kips}$$

$\phi V_n = 0.75\,(145.3) = 109.0 \text{ kips} > V_u = 86.4 \text{ kips}$ O.K.

4. Determine shear-friction reinforcement A_{vf}. *11.9.3.2*

Using a Modified Shear-Friction Method as permitted by 11.7.3 (see R11.7.3):

$V_n = 0.8A_{vf}f_y + K_1 b_w d$, with $\frac{A_{vf}f_y}{b_w d}$ not less than 200 psi

For all lightweight concrete, K_1 = 200 psi *R11.7.3*

$V_u \le \phi V_n = \phi\,(0.8A_{vf}f_y + 0.2b_w d)$

Solving for A_{vf}:

$$A_{vf} = \frac{V_u - \phi(0.2b_w d)}{\phi(0.8f_y)}, \text{ but not less than } 0.2 \times \frac{b_w d}{f_y}$$

$$= \frac{86.4 - (0.75 \times 0.2 \times 14 \times 14)}{0.75\,(0.8 \times 60)} = 1.58 \text{ in.}^2 \text{ (governs)}$$

but not less than $0.2 \times \frac{b_w d}{f_y} = 0.2 \times \frac{14 \times 14}{60} = 0.65 \text{ in.}^2$

For comparison, compute A_{vf} by Eq. (11-25): *11.7.4.3*

For lightweight concrete,

$\mu = 1.4\lambda = 1.4\,(0.75) = 1.05$

$$A_{vf} = \frac{V_u}{\phi f_y \mu} = \frac{86.4}{0.75 \times 60 \times 1.05} = 1.83 \text{ in.}^2 > 1.58 \text{ in.}^2$$

Note: Modified shear-friction method presented in R11.7.3 would give a closer estimate of shear-transfer strength than the conservative shear-friction method in 11.7.4.1.

5. Determine flexural reinforcement A_f. *11.9.3.3*

$M_u = V_u a_v + N_{uc}\,(h - d) = 86.4\,(3) + 38.4\,(15 - 14.0) = 297.6$ in.-kips

Find A_f using conventional flexural design methods, or conservatively use $j_u d = 0.9d$

$$A_f = \frac{M_u}{\phi f_y j_u d} = \frac{297.6}{0.75 \times 60 \times 0.9 \times 14} = 0.53 \text{ in.}^2$$

Note that for all design calculations, $\phi = 0.75$ *11.9.3.1*

6. Determine direct tension reinforcement A_n. *11.9.3.4*

$$A_n = \frac{N_{uc}}{\phi f_y} = \frac{38.4}{0.75 \times 60} = 0.85 \text{ in.}^2$$

7. Determine primary tension reinforcement A_{sc}. *11.9.3.5*

$\left(\frac{2}{3}\right)A_{vf} = \left(\frac{2}{3}\right) 1.83 = 1.22 \text{ in.}^2 > A_f = 0.53 \text{ in.}^2$; Therefore, $\left(\frac{2}{3}\right)A_{vf}$ controls design.

$$A_{sc} = \left(\frac{2}{3}\right)A_{vf} + A_n = 1.22 + 0.85 = 2.07 \text{ in.}^2$$

Use 3-No. 8 bars, $A_{sc} = 2.37 \text{ in.}^2$

Check $A_{sc(min)} = 0.04 \left(\frac{4}{60}\right) 14 \times 14 = 0.52 \text{ in.}^2 < A_{sc} = 2.37 \text{ in.}^2$ O. K. *11.9.5*

8. Determine shear reinforcement A_h. *11.9.4*

$A_h = 0.5 (A_{sc} - A_n) = 0.5 (2.37 - 0.85) = 0.76 \text{ in.}^2$

Use 4-No. 3 stirrups, $A_h = 0.88 \text{ in.}^2$

The shear reinforcement is to be placed within two-thirds of the effective corbel depth adjacent to A_{sc}.

$s_{max} = \left(\frac{2}{3}\right)\frac{14}{4} = 2.33$ in. Use 2¼ in. o.c. stirrup spacing.

9. Corbel details

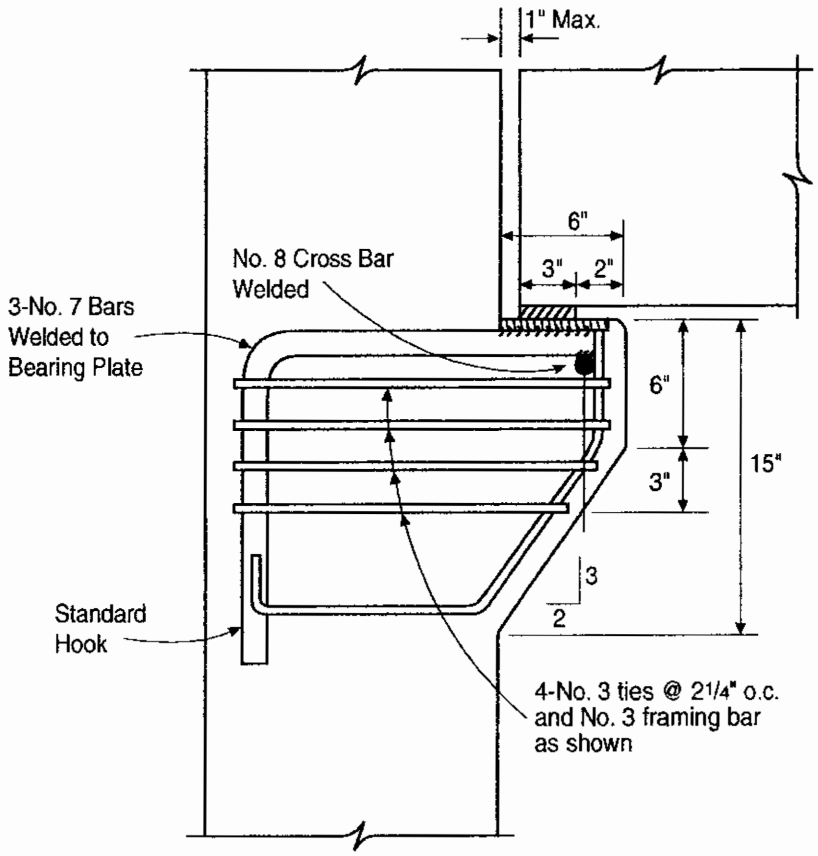
Corbel will project (1 + 3 + 2) = 6 in. from column face.

Use 6-in. depth at outer face of corbel, then depth at outer edge of bearing plate will be

$6 + 3 = 9 \text{ in.} > \frac{14}{2} = 7.0$ in. O.K. *11.9.2*

A_{sc} to be anchored at front face of corbel by welding a No. 8 bar transversely across ends of A_{sc} bars. *11.9.6*

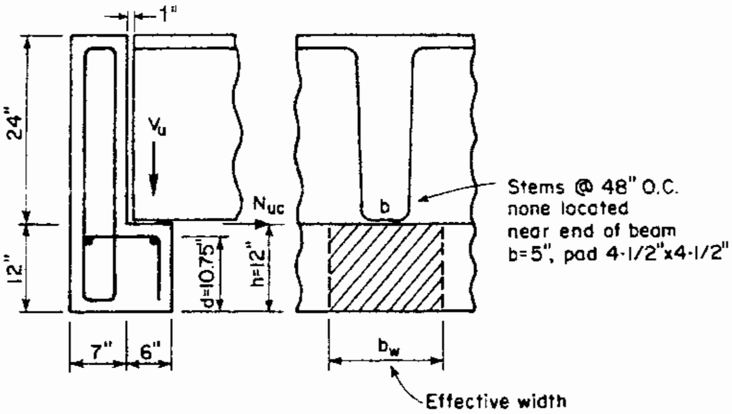
A_{sc} must be anchored within column by standard hook.

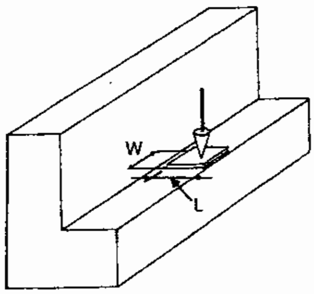

1" Max.
6"
3"
2"
No. 8 Cross Bar
Welded
3-No. 7 Bars
Welded to
Bearing Plate
6"
3"
15"
3
2
Standard
Hook
4-No. 3 ties @ 2 1/4" o.c.
and No. 3 framing bar
as shown

Example 15.3—Beam Ledge Design

f'_c = 5000 psi (normal weight)

f_y = 60,000 psi





The L-beam shown is to support a double-tee parking deck spanning 64 ft. Maximum service loads per stem are: DL = 11.1 kips; LL = 6.4 kips; total load = 17.5 kips. The loads may occur at any location on the L-beam ledge except near beam ends. The stems of the double-tees rest on 4.5 in. × 4.5 in. × 1/4 in. neoprene bearing pads (1000 psi maximum service load).

Design in accordance with the code provisions for brackets and corbels may require a wider ledge than the 6 in. shown. To maintain the 6-in. width, one of the following may be necessary: (1) Use of a higher strength bearing pad (up to 2000 psi); or (2) Anchoring primary ledge reinforcement A_{sc} to an armor angle.

This example will be based on the 6-in. ledge with 4.5-in.-square bearing pad. At the end of the example an alternative design will be shown.

Note: This example illustrates design to prevent potential local failure modes. In addition, ledge beams should be designed for global effects, not considered in this example. For more details see References 15.2 to 15.6.

Calculations and Discussion	Code Reference

1. Check 4.5 × 4.5 in. bearing pad size (1000 psi maximum service load).

   Capacity = 4.5 × 4.5 × 1.0 = 20.3 kips > 17.5 kips O.K.

2. Determine shear spans and effective widths for both shear and flexure [Ref. 15.3 to 15.5]. The reaction is considered at outer third point of the bearing pad.

   a. For shear friction

   $$a_v = 4.5\left[\frac{2}{3}\right] + 1.0 = 4 \text{ in.}$$

   Effective width = $W + 4a_v$ = 4.5 + 4 (4) = 20.5 in.

b. For flexure, critical section is at center of the hanger reinforcement (A_v)

Assume 1 in. cover and No. 4 bar stirrups

$a_f = 4 + 1 + 0.25 = 5.25$ in.

Effective width $= W + 5a_f = 4.5 + 5\,(5.25) = 30.75$ in.

3. Check concrete bearing strength.

$V_u = 1.2\,(11.1) + 1.6\,(6.4) = 23.6$ kips — *Eq. (9-2)*

$\phi P_{nb} = \phi(0.85 f'_c A_1)$ — *10.17.1*

$\phi = 0.65$ — *9.3.2.4*

$\phi P_{nb} = 0.65\,(0.85 \times 5 \times 4.5 \times 4.5) = 55.9$ kips > 23.6 kips O.K.

4. Check effective ledge section for maximum nominal shear-transfer strength V_n. — *11.9.3.2.1*

For $f'_c = 5000$ psi: $V_n(\max) = 800 b_w d$, where $b_w = (W + 4a_v) = 20.5$ in.

$$V_n = \frac{800\,(20.5)\,(10.75)}{1000} = 176.3 \text{ kips}$$

$\phi = 0.75$ — *11.9.3.1*

$\phi V_n = 0.75\,(176.3) = 132.2$ kips > 23.6 kips O.K.

5. Determine shear-friction reinforcement A_{vf}. — *11.9.3.2*

$$A_{vf} = \frac{V_u}{\phi f_y \mu} = \frac{23.6}{0.75\,(60)\,1.4} = 0.37 \text{ in.}^2\text{/per effective width of 20.5 in.}$$ — *11.7.4.1*

where $\mu = 1.4$ — *11.7.4.3*

6. Check for punching shear (Eq. (3))

$V_u \le 4\phi\sqrt{f'_c}\,(W + 2L + 2d_f)\,d_f$

$W = L = 4.5$ in.

$d_f \approx 10$ in. (assumed)

$$4\phi\sqrt{f'_c}\,(3W + 2d_f)\,d_f = 4 \times 0.75 \times \sqrt{5000}\,[(3 \times 4.5) + (2 \times 10)] \times 10/1000$$

$$= 71.1 \text{ kips} > 23.6 \text{ kips}$$

7. Determine reinforcement to resist direct tension A_n. Unless special provisions are made to reduce direct tension, N_u should be taken not less than $0.2V_u$ to account for unexpected forces due to restrained long-time deformation of the supported member, or other causes. When the beam ledge is designed to resist specific horizontal forces, the bearing plate should be welded to the tension reinforcement A_{sc}. *11.9.3.4*

$N_u = 0.2V_u = 0.2\ (23.6) = 4.7$ kips

$$A_n = \frac{N_u}{\phi f_y} = \frac{4.7}{0.75\ (60)} = 0.10\ \text{in.}^2\text{/per effective width of 30.75 in. (0.003 in.}^2\text{/in.)}$$

8. Determine flexural reinforcement A_f.

$M_u = V_u a_f + N_u (h - d) = 23.6\ (5.25) + 4.7\ (12 - 10.75) = 129.8$ in.-kips

Find A_f using conventional flexural design methods. For beam ledges, Ref. 15.5 recommends to use $j_u d = 0.8d$. *11.9.3.3*

$\phi = 0.75$ *11.9.3.1*

$$A_f = \frac{129.8}{0.75\ (60)\ (0.8 \times 10.75)} = 0.34\ \text{in.}^2\text{/per 30.75 in. width} = 0.011\ \text{in.}^2\text{/in.}$$

9. Determine primary tension reinforcement A_{sc}. *11.9.3.5*

$$\left(\frac{2}{3}\right)A_{vf} = \left(\frac{2}{3}\right)0.37 = 0.25\ \text{in.}^2\text{/per 20.5 in. width} = 0.012\ \text{in.}^2\text{/in.}$$

$$A_{sc} = \left(\frac{2}{3}\right)A_{vf} + A_n = 0.012 + 0.003 = 0.015\ \text{in.}^2\text{/in. (governs)}$$

$$A_{sc} = A_f + A_n = 0.011 + 0.003 = 0.014\ \text{in.}^2\text{/in.}$$

Check $A_{sc(min)} = 0.04\left(\dfrac{f'_c}{f_y}\right)d$ per in. width *11.9.5*

$$= 0.04\left(\frac{5}{60}\right)10.75 = 0.036\ \text{in.}^2\text{/in.} > 0.015\ \text{in.}^2\text{/in.}$$

For typical shallow ledge members, minimum A_{sc} by 11.9.5 will almost always govern.

10. Determine shear reinforcement A_h.

$A_h = 0.5\ (A_{sc} - A_n) = 0.5\ (0.036 - 0.003) = 0.017$ in.^{2}/in. *11.9.4*

11. Determine final size and spacing of ledge reinforcement.

For A_{sc} = 0.036 in.^{2}/in.:

Try No. 5 bars (A = 0.31 in.^{2})

$$s_{max} = \frac{0.31}{0.036} = 8.6 \text{ in.}$$

Use No. 5 @ 8 in.

A_h = 0.017 in.^{2}/in. For ease of constructability, provide reinforcement A_h at same spacing of 8 in.

Provide No. 4 (A = 0.2 in.^{2}) @ 8 in. within 2/3d adjacent to A_{sc}.

12. Check required area of hanger reinforcement.

For strength (Eq. (4)):

$$A_v = \frac{V_u s}{\phi f_y S}$$

For s = 8 in. and S = 48 in.

$$A_v = \frac{23.6 \times 8}{0.75 \times 60 \times 48} = 0.09 \text{ in.}^2$$

For serviceability (Eq. (5)):

$$A_v = \frac{V}{0.5f_y} \times \frac{s}{(W + 3a_v)}$$

V = 11.1 + 6.4 = 17.5 kips

$W + 3a_v$ = 4.5 + (3 × 4) = 16.5 in.

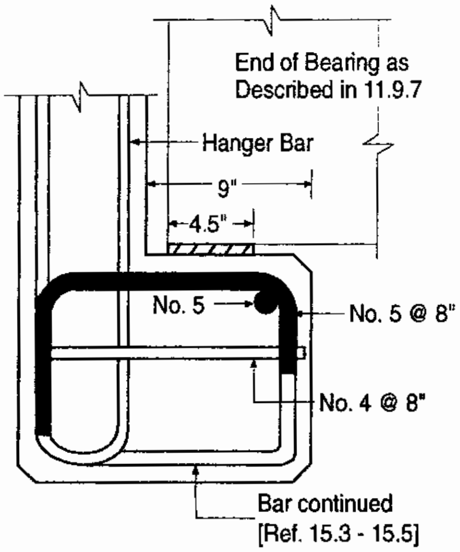
$$A_v = \frac{17.5}{0.5 \times 60} \times \frac{8}{16.5} = 0.28 \text{ in.}^2 \text{ (governs)}$$

No. 5 hanger bars @ 8 in. are required

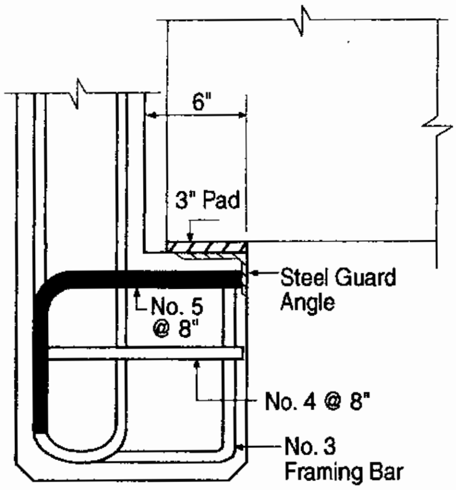
Sufficient stirrups for combined shear and torsion must be provided for global effects in the ledge beam. (See Refs. 15.5 and 15.6)

13. Reinforcement Details

In accordance with 11.9.7, bearing area (4.5 in. pad) must not extend beyond straight portion of beam ledge reinforcement, nor beyond inside edge of transverse anchor bar. With a 4.5 in. bearing pad, this requires that the width of ledge be increased to 9 in. as shown below. Alternately, a 6 in. ledge with a 3 in. medium strength pad (1500 psi) and the ledge reinforcement welded to an armor angle would satisfy the intent of 11.9.7.



9 in. Ledge Detail



6 in. Ledge Detail
(Alternate)

16

Shear in Slabs

UPDATE FOR THE '05 CODE

The expression $2\sqrt{f'_c}$ is replaced by $\phi\left(2\sqrt{f'_c}\right)$ in 11.12.6.2 to correct a typographical error.

11.12 SPECIAL PROVISIONS FOR SLABS AND FOOTINGS

The provisions of 11.12 must be satisfied for shear design in slabs and footings. Included are requirements for critical shear sections, nominal shear strength of concrete, and shear reinforcement.

11.12.1 Critical Shear Section

In slabs and footing, shear strength in the vicinity of columns, concentrated loads, or reactions is governed by the more severe of two conditions:

- Wide-beam action, or one-way shear, as evaluated by provisions 11.1 through 11.5.
- Two-way action, as evaluated by 11.12.2 through 11.12.6.

Analysis for wide-beam action considers the slab to act as a wide beam spanning between columns. The critical section extends in a plane across the entire width of the slab and is taken at a distance d from the face of the support (11.12.1.1); see Fig. 16-1. In this case, the provisions of 11.1 through 11.5 must be satisfied. Except for long, narrow slabs, this type of shear is seldom a critical factor in design, as the shear force is usually well below the shear capacity of the concrete. However, it must be checked to ensure that shear strength is not exceeded.

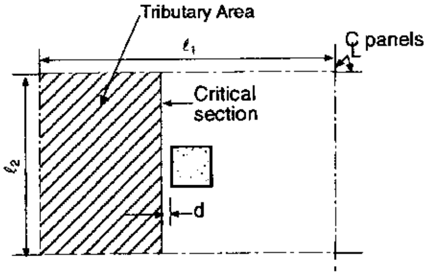


Fig. 16-1 Tributary Area and Critical Section for Wide-Beam Shear

Two-way or "punching" shear is generally the more critical of the two types of shear in slab systems supported directly on columns. Depending on the location of the column, concentrated load, or reaction, failure can occur along two, three, or four sides of a truncated cone or pyramid. The perimeter of the critical section b_o is located in such a manner that it is a minimum, but need not approach closer than a distance d / 2 from edges or corners of columns, concentrated loads, or reactions, or from changes in slab thickness such as edges of capitals or drop panels (11.12.1.2); see Fig. 16-2. In this case the provisions of 11.12.2 through 11.12.6 must be satisfied. It is important to note that it is permissable to use a rectangular perimeter b_o to define the critical section for square or rectangular columns, concentrated loads, or reaction areas (11.12.1.3).

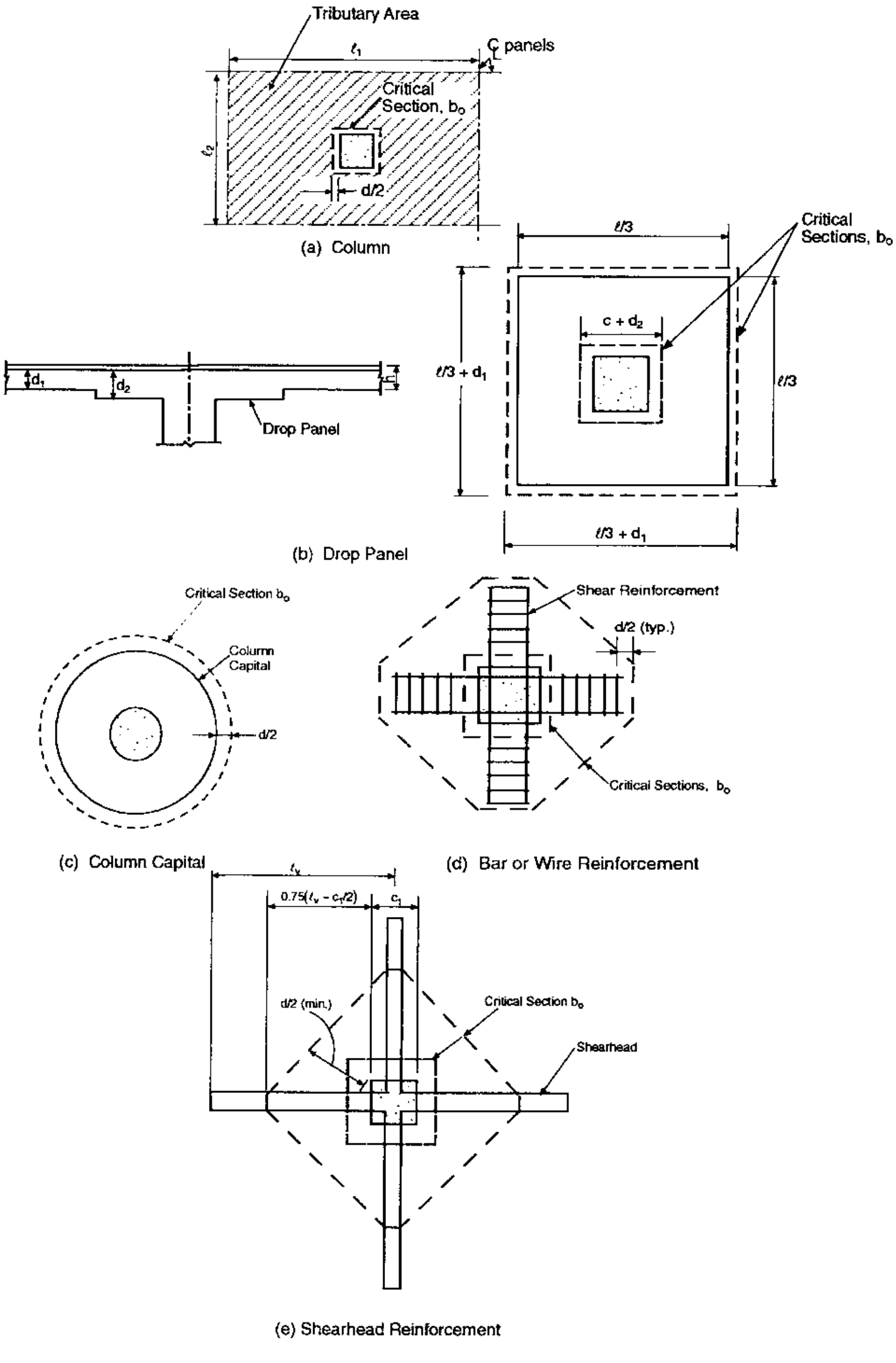


Fig. 16-2 Tributary Areas and Critical Sections for Two-Way Shear

11.12.2 Shear Strength Requirement for Two-Way Action

In general, the factored shear force V_u at the critical shear section shall be less than or equal to the shear strength ϕV_n:

$$\phi V_n \geq V_u \qquad \textit{Eq. (11-1)}$$

where the nominal shear strength V_n is:

$$V_n = V_c + V_s \qquad \textit{Eq. (11-2)}$$

and

V_c = nominal shear strength provided by concrete, computed in accordance with 11.12.2.1 if shear reinforcement is not used or 11.12.3.1 if shear reinforcement is used.

V_s = nominal shear strength provided by reinforcement, if required, computed in accordance with 11.12.3 if bars, wires, or stirrups are used, or 11.12.4 if shearheads are used. Where moment is transferred between the slab and the column in addition to direct shear, 11.12.6 shall apply.

11.12.2.1 Nominal shear strength provided by concrete V_c for slabs without shear reinforcement

The shear stress provided by concrete at a section v_c is a function of the concrete compressive stress f'_c, and is limited to $4\sqrt{f'_c}$ for square columns. The nominal shear strength provided by concrete V_c is obtained by multiplying v_c by the area of concrete section resisting shear transfer, which is equal to the perimeter of the critical shear section b_o multiplied by the effective depth of the slab d:

$$V_c = 4\sqrt{f'_c}\,b_o d \qquad \textit{Eq. (11-35)}$$

Tests have indicated that the value of $4\sqrt{f'_c}$ is unconservative when the ratio of the long and short sides of a rectangular column or loaded area β_c is larger than 2.0. In such cases, the shear stress on the critical section varies as shown in Fig. 16-3. Equation (11-33) accounts for the effect of β_c on the concrete shear strength:

$$V_c = \left(2 + \frac{4}{\beta}\right)\sqrt{f'_c}\,b_o d \qquad \textit{Eq. (11-33)}$$

From Fig. 16-3, it can be seen that for $\beta \leq 2.0$ (i.e., square or nearly square column or loaded area), two-way shear action governs, and the maximum concrete shear stress v_c is $4\sqrt{f'_c}$. For values of β value larger than 2.0, the concrete stress decreases linearly to a minimum $2\sqrt{f'_c}$, which is equivalent to shear stress for one-way shear.

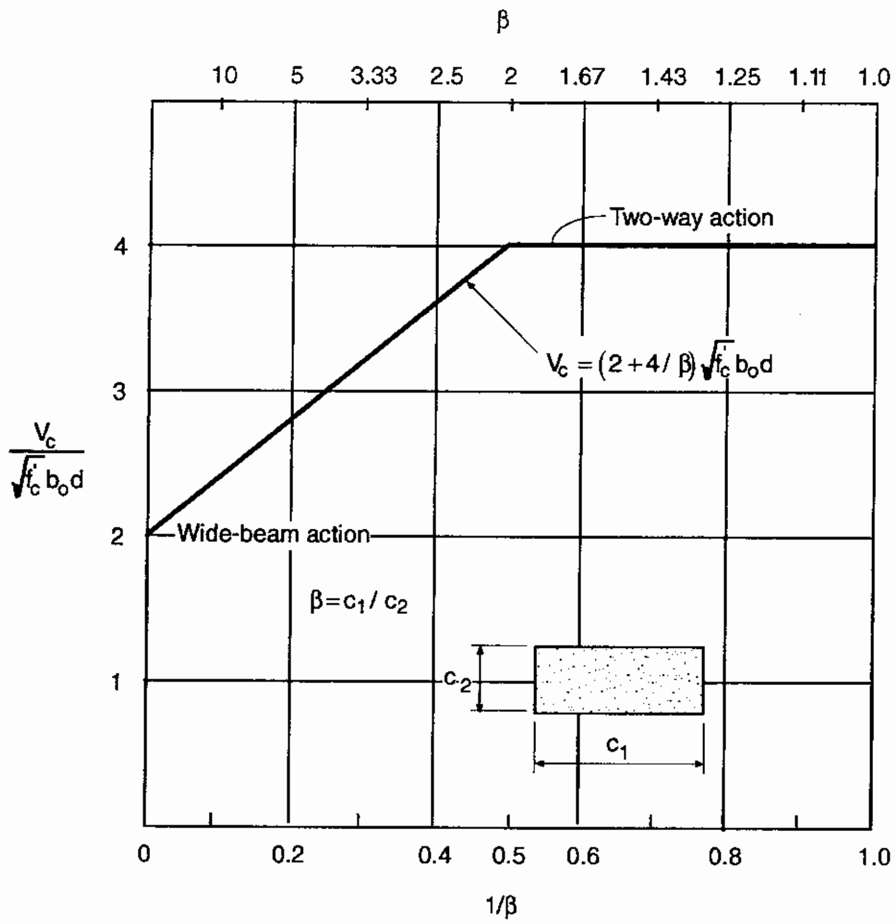


Fig. 16-3 Effect of β on Concrete Shear Strength

Other tests have indicated that v_c decreases as the ratio b_o/d increases. Equation (11-34) accounts for the effect of b_o/d on the concrete shear strength:

$$V_c = \left(\frac{\alpha_s d}{b_o} + 2\right)\sqrt{f'_c}\, b_o d \qquad \textit{Eq. (11-34)}$$

Figure 16-4 illustrates the effect of b_o/d for interior, edge, and corner columns, where α_s equals 40, 30, and 20, respectively. For an interior column with $b_o/d \le 2.0$, the maximum permissible shear stress is $4\sqrt{f'_c}$; see Fig. 16-4. Once $b_o/d > 2.0$, the shear stress decreases linearly to $2\sqrt{f'_c}$ at b_o/d equal to infinity.

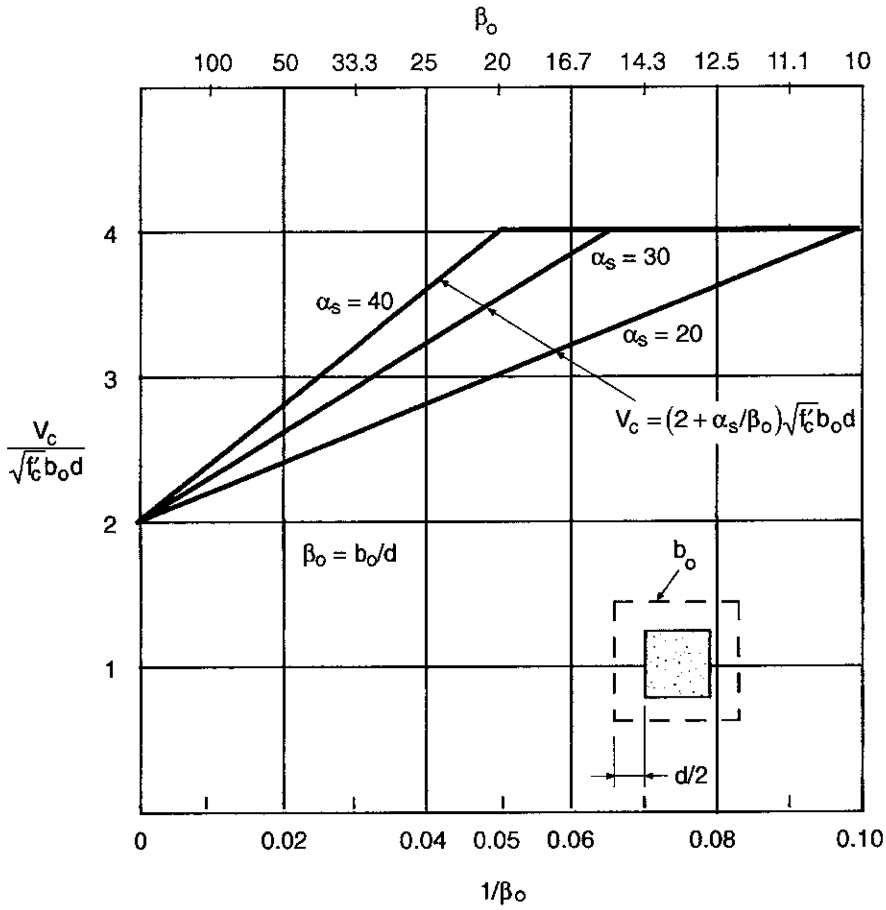


Fig. 16-4 Effect of b_o / d on Concrete Shear Strength

Note that reference to interior, edge, and corner column does not suggest column location in a building, but rather refers to the number of sides of the critical section available to resist the shear stress. For example, a column that is located in the interior of a building, with one side at the edge of an opening, shall be evaluated as an edge column.

The concrete nominal shear strength for two-way shear action of slabs without shear reinforcement is the least of Eqs. (11-33), (11-34), and (11-35) (11.12.2.1). Note that if lightweight concrete is used, 11.2 shall apply.

11.12.3 Shear Strength Provided by Bars, Wires, and Single or Multiple-Leg Stirrups

The use of bars, wires, or single or multiple-leg stirrups as shear reinforcement in slabs is permitted provided that the effective depth of the slab is greater than or equal to 6 inches, but not less than 16 times the shear reinforcement bar diameter (11.12.3). Suggested rebar shear reinforcement consist of properly anchored single-leg, multiple-leg, or closed stirrups that are engaging longitudinal reinforcement at both the top and bottom of the slab (11.12.3.4); see Fig. R11.12.3 (a), (b), (c).

With the use of shear reinforcement, the nominal shear strength provided by concrete V_c shall not be taken greater than $2\sqrt{f'_c}b_od$ (11.12.3.1), and nominal shear strength V_n is limited to $6\sqrt{f'_c}b_od$ (11.12.3.2). Thus, V_s must not be greater than $4\sqrt{f'_c}b_od$.

The area of shear reinforcement A_v is computed from Eq. (11-15), and is equal to the cross-sectional area of all legs of reinforcement on one peripheral line that is geometrically similar to the perimeter of the column section (11.12.3.1):

$$A_v = \frac{V_s s}{f_y d} \qquad \textit{Eq. (11-15)}$$

The spacing limits of 11.12.3.3 correspond to slab shear reinforcement details that have been shown to be effective. These limits are as follows (see Fig. 16-5):

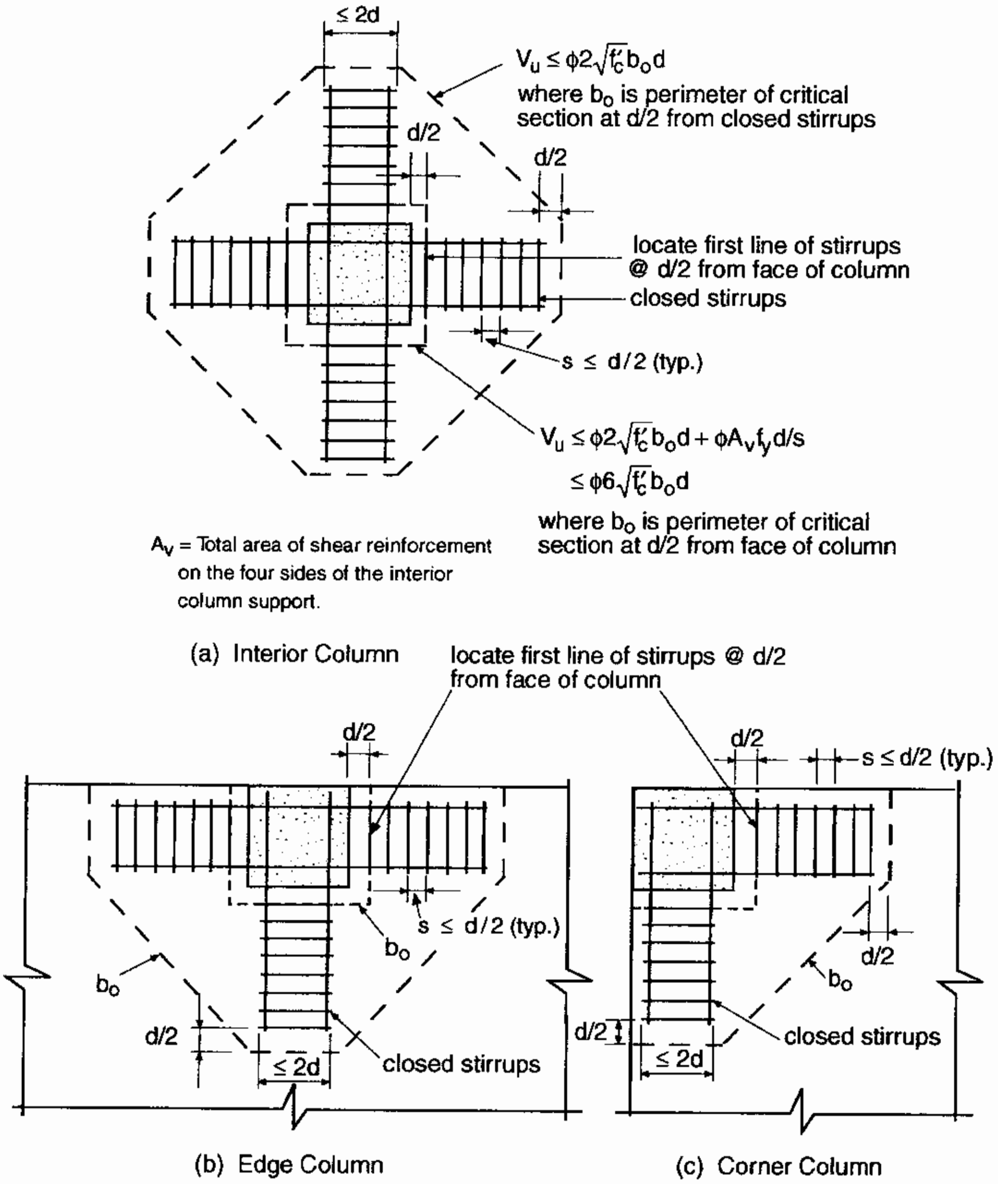


Fig. 16-5 Design and Detailing Criteria for Slabs with Stirrups

1. The first line of stirrups surrounding the column shall be placed at distance not exceeding d/2 from the column face.
2. The spacing between adjacent legs in the first line of shear reinforcement shall not exceed 2d.
3. The spacing between successive lines of shear reinforcement that surround the column shall not exceed d / 2.
4. The shear reinforcement can be terminated when $V_u \le \phi 2\sqrt{f'_c} b_o d$ (11.12.3.1).

Proper anchorage of the shear reinforcement is achieved by satisfying the provisions of 12.13 (11.12.3.4). Refer to Fig. R11.12.3 and Part 4 for additional details on stirrup anchorage. It should be noted that anchorage requirements of 12.13 may be difficult for slabs thinner than 10 inches. Application of shear reinforcement design using bars or stirrups is illustrated in Example 16.3.

Where moment transfer is significant between the column and the slab, it is recommended to use closed stirrups in a pattern as symmetrical as possible around the column (R11.12.3).

11.12.4 Shear Strength Provided by Shearheads

The provisions of 11.12.4 permit the use of structural steel sections such as I- or channel-shaped sections (shearheads) as shear reinforcement in slabs, provided the following criteria are satisfied:

1. Each arm of the shearhead shall be welded to an identical perpendicular arm with full penetration welds and each arm must be continuous within the column section (11.12.4.1); see Fig. 16-6 (a).
2. Shearhead depth shall not exceed 70 times the web thickness of the steel shape (11.12.4.2); see Fig. 16-6 (b).
3. Ends of each shearhead arm is permitted to be cut at angles not less than 30 deg with the horizontal, provided the tapered section is adequate to resist the shear force at that location (11.12.4.3); see Fig. 16-6 (b).
4. All compression flanges of steel shapes shall be located within 0.3d of compression surface of slab, which in the case of direct shear, is the distance measured from the bottom of the slab (11.12.4.4); see Fig. 16-6 (b).
5. The ratio α_v of the flexural stiffness of the steel shape to surrounding composite cracked slab section of width $c_2 + d$ shall not be less than 0.15 (11.12.4.5); see Fig. 16-6 (c).
6. The required plastic moment strength M_p is computed from the following equation (11.12.4.6):

$$\phi M_p = \frac{V_u}{2n}\left[h_v + \alpha_v(\ell_v - 0.5c_1)\right] \qquad \textit{Eq. (11-37)}$$

where:

M_p = plastic moment strength for each shearhead arm required to ensure that the ultimate shear is attained as the moment strength of the shearhead is reached.

ϕ = strength reduction factor for tension-controlled members, equal to 0.9 per 9.3.2.3.

n = number of shearhead arms; see Fig. R11.12.4.7.

ℓ_v = minimum required length of shearhead arm per 11.12.4.7 and 11.12.4.8; see Fig. R11.12.4.7.

h_v = depth of shearhead cross-section; see Fig. 16-6 (b).

7. The critical section for shear shall be perpendicular to the plane of the slab and shall cross each shearhead arm at three-quarters of the distance $(\ell_v - 0.5c_1)$ from the column face to the end of the shearhead arm. The critical section shall be located per 11.12.1.2(a) (11.12.4.7); see Fig. R11.12.4.7.

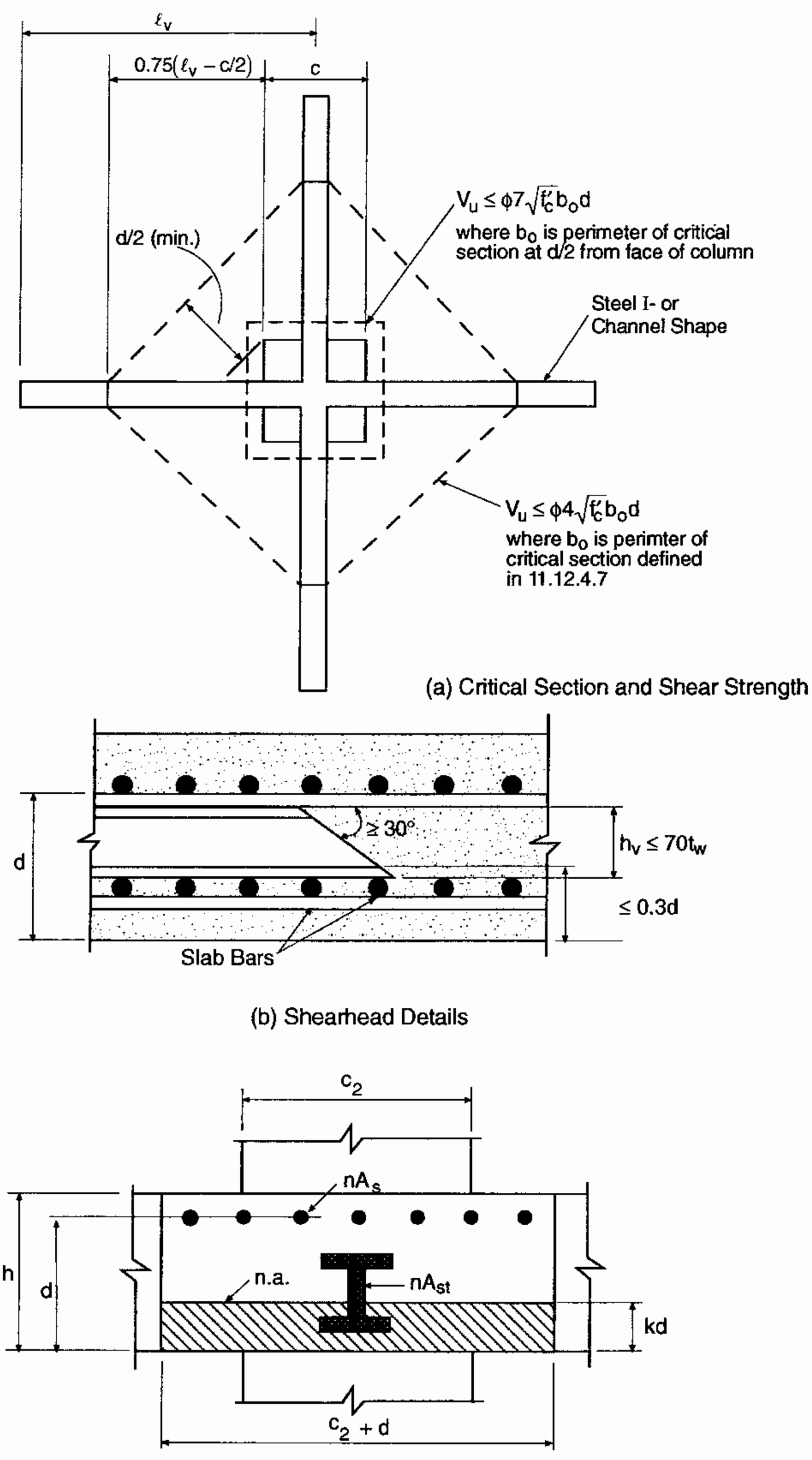


Fig. 16-6 Design and Detailing Criteria for Slabs with Shearhead Reinforcement

8. The nominal shear strength V_n shall be less than or equal to $4\sqrt{f'_c}b_o d$ on the critical section defined by 11.12.4.7, and $7\sqrt{f'_c}b_o d$ at d / 2 distance from the column face (11.12.4.8); see Fig. 16-6 (a).

9. Section 11.12.4.9 permits the shearheads to contribute in resisting the slab design moment in the column strip. The moment resistance M_v contributed to each column strip shall be the minimum of:

   a. $$\frac{\phi \alpha_v V_u}{2n}(\ell_v - 0.5c_1) \qquad \textit{Eq.(11-38)}$$

   b. $0.30M_u$ of the total factored moment in each column strip
   c. the change in column strip moment over the length ℓ_v
   d. the value of M_p computed by Eq. (11-37).

When direct shear and moment are transferred between slab and column, the provisions of 11.12.6 must be satisfied in addition to the above criteria. In slabs with shearheads, integrity steel shall be provided in accordance with 13.3.8.6. Application on the design of shearheads as shear reinforcement is illustrated in Example 16.3.

Other Type of Shear Reinforcement

Slab shear reinforcement consisting of vertical bars mechanically anchored at each end by a plate or a head capable of developing the yield strength of the bars have been used successfully (R11.12.3); see Fig. 16-7. This type of shear reinforcement for slabs can be advantageous when considering their ease of installation and the cost of placement, compared to other types of slab shear reinforcement. Extensive tests, methods of design, and worked-out design examples are presented in Refs. 16.1 through 16.4.

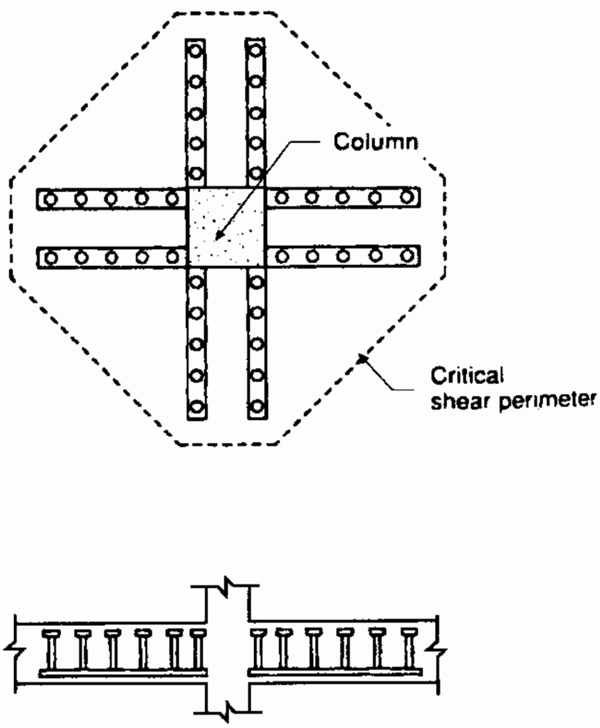


Fig. 16-7 Shear Reinforcement by Headed Studs

11.12.5 Effect of Openings in Slabs on Shear Strength

The effect of openings in slabs on concrete shear strength shall be considered when the opening is located: (1) anywhere within a column strip of a flat slab system and (2) within 10 times the slab thickness from a concentrated load or reaction area. Slab opening effect is evaluated by reducing the perimeter of the critical section b_o by a length equal to the projection of the opening enclosed by two-lines extending from the centroid of the column and tangent to the opening; see Fig 16-8 (a). For slabs with shear reinforcement, the ineffective portion of the perimeter b_o is one-half of that without shear reinforcement; see Fig. 16-8 (b). The one-half factor is interpreted to apply equally to shearhead reinforcement and bar or wire reinforcement as well. Effect of opening in slabs on flexural strength is discussed in Part 18.

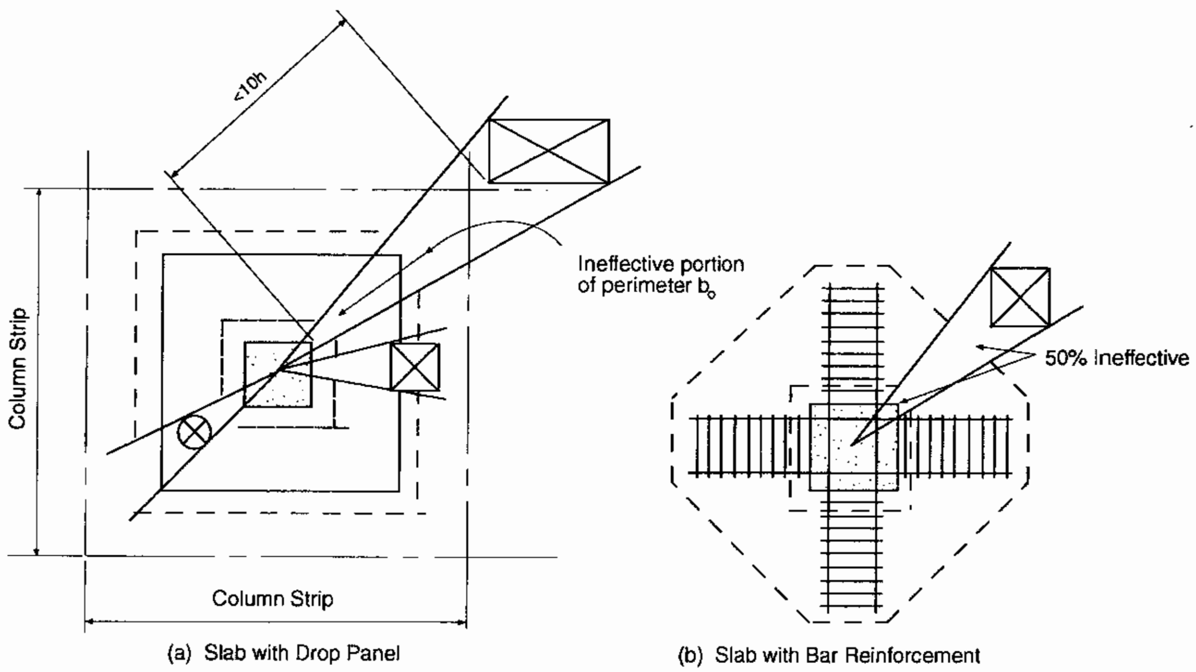


Fig. 16-8 Effect of Openings in Slabs on Shear Strength

11.12.6 Moment Transfer at Slab-Column Connections

For various loading conditions, unbalanced moment M_u can occur at the slab-column connections. For slabs without beams between supports, the transfer of unbalanced moment is one of the most critical design conditions for two-way slab systems. Shear strength at an exterior slab-column connection (without spandrel beam) is especially critical, because the total exterior negative moment must be transferred to the column, which is in addition to the direct shear due to gravity loads; see Fig. 16-9. The designer should not take this aspect of two-way slab design lightly. Two-way slab systems usually are fairly "forgiving" in the event of an error in the amount and or distribution of flexural reinforcement; however, little or no forgiveness is to be expected if shear strength provisions are not fully satisfied.

Note that the provisions of 11.12.6 (or 13.5.3) do not apply to slab systems with beams framing into the column support. When beams are present, load transfer from the slab through the beams to the columns is considerably less critical. Shear strength in slab systems with beams is covered in 13.6.8.

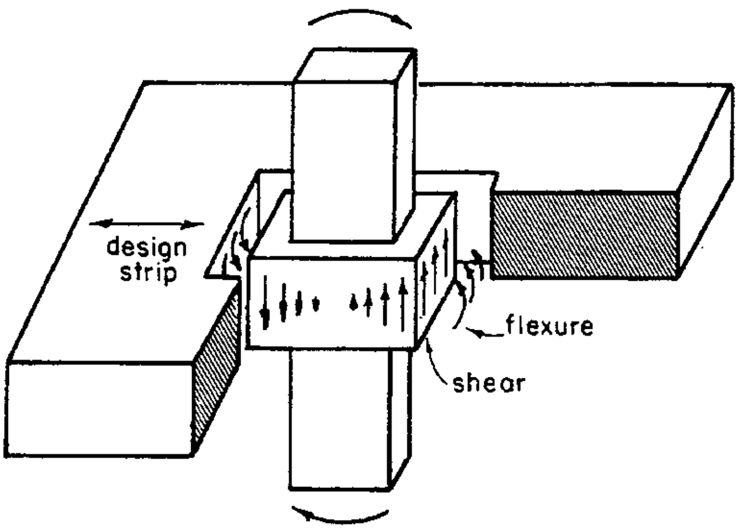


Fig. 16-9 Direct Shear and Moment Transfer

11.12.6.1 Distribution of Unbalanced Moment

The code specifies that the unbalanced moment at a slab-column connection must be transferred from the slab (without beams) to the column by eccentricity of shear in accordance with 11.12.6 and by flexure in accordance with 13.5.3 (11.12.6.1). Studies (Ref. 16.7) of moment transfer between slabs and square columns found that $0.6M_u$ is transferred by flexure across the perimeter of the critical section b_o defined by 11.12.1.2, and $0.4M_u$ by eccentricity of shear about the centroid of the critical section. For a rectangular column, the portion of moment transferred by flexure $\gamma_f M_u$ increases as the dimension of the column that is parallel to the applied moment increases. The fraction of unbalanced moment transferred by flexure γ_f is:

$$\gamma_f = \frac{1}{1+\left(\frac{2}{3}\right)\sqrt{\frac{b_1}{b_2}}} \qquad \textit{Eq. (13-1)}$$

and the fraction of unbalanced moment transferred by eccentricity of shear is:

$$\gamma_v = 1 - \gamma_f \qquad \textit{Eq. (11-39)}$$

where b_1 and b_2 are the dimensions of the perimeter of the critical section, with b_1 parallel to the direction of analysis; see Fig. 16-10. The relationship of the parameters presented into Eqs. (13-1) and (11-39) is graphically illustrated in Fig. 16-11. Modification or adjustment of γ_f and thus γ_v, is permitted in accordance with 13.5.3.3 for any two-way slab system, except for prestressed slabs. The following modifications are applicable, provided that the reinforcement ratio in the slab within the effective width defined in 13.5.3.2 does not exceed $0.375\rho_b$:

- For unbalanced moments about an axis parallel to the slab edge at exterior supports (i.e., bending perpendicular to the edge), it is permitted to take $\gamma_f = 1.0$ provided that $V_u \le 0.75\phi V_c$ at an edge column or $V_u \le 0.5\phi V_c$ at a corner column.
- For unbalanced moments at interior supports and for unbalanced moments about an axis transverse to the edge of exterior supports (i.e., bending parallel to the edge), it is permitted to increase γ_f by up to 25%, provided that $V_u \le 0.4\phi V_c$.

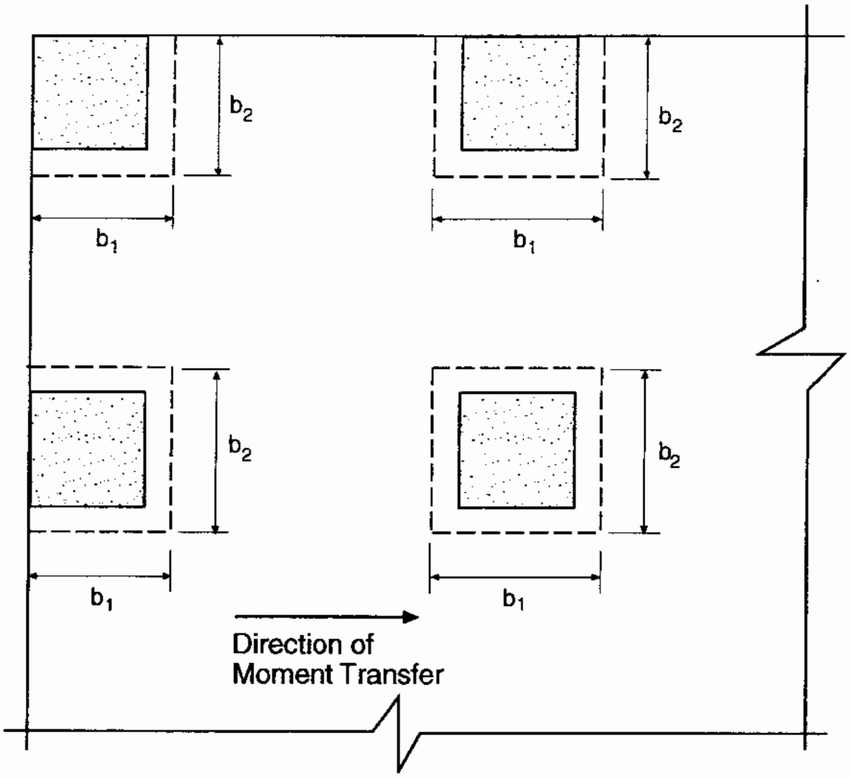


Fig. 16-10 Parameters b_1 and b_2 for Eqs. (11-39) and (13-1)

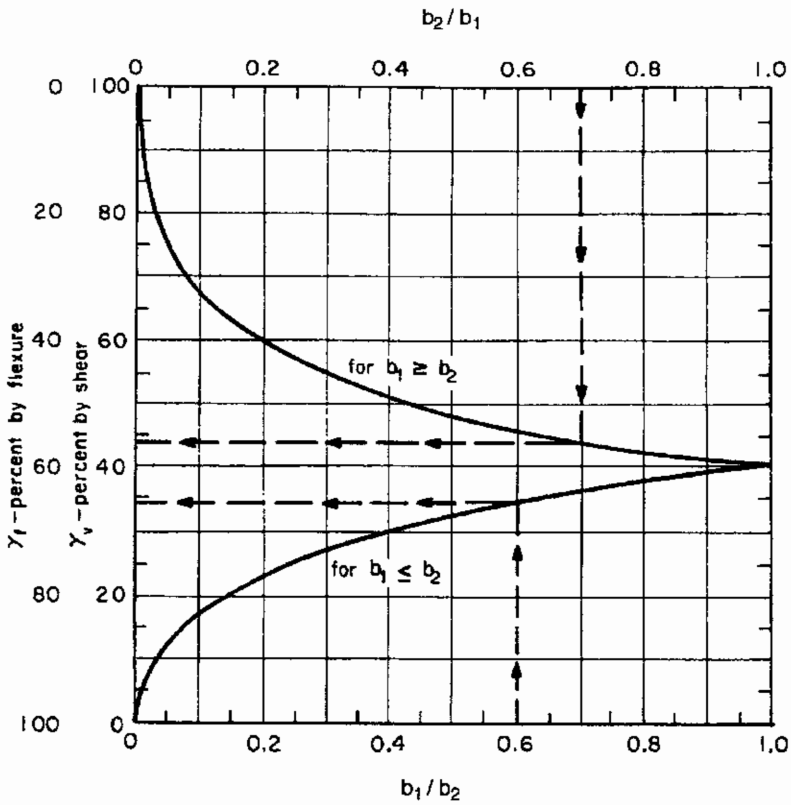


Fig. 16-11 Graphical Solution of Eqs. (13-1) and (11-39)

The unbalanced moment transferred by eccentricity of shear is $\gamma_v M_u$, where M_u is the unbalanced moment at the centroid of the critical section. The unbalanced moment M_u at an exterior support of an end span will generally not be computed at the centroid of the critical transfer section in the frame analysis. When the Direct Design Method of Chapter 13 is utilized, moments are computed at the face of the support. Considering the approximate nature of the procedure to evaluate the stress distribution due to moment-shear transfer, it seems unwarranted to consider a change in moment to the transfer centroid; use of the moment values from frame analysis (centerline of support) or from 13.6.3.3 (face of support) is accurate enough.

Unbalanced moment transfer between an edge column and a slab without edge beams requires special consideration when slabs are analyzed for gravity loads using the moment coefficients of the Direct Design Method. In this case, unbalanced moment M_u must be set equal to $0.3M_o$ (13.6.3.6), where M_o is the total factored static moment in the span. Therefore, the fraction of unbalanced moment transferred by shear is $\gamma_v M_u = \gamma_v(0.3M_o)$. See Part 19 for further discussion of that special shear strength requirement and its application in Example 19.1. If the Equivalent Frame Method is used, the unbalanced moment is equal to the computed frame moment.

11.12.6.2 Shear Stresses and Strength Computation

Assuming that shear stress resulting from moment transfer by eccentricity of shear varies linearly about the centroid of the critical section defined in 11.12.1.2, the factored shear stresses at the faces of the critical section due to the direct shear V_u and the unbalanced moment transferred by eccentricity of shear $\gamma_v M_u$ are (see Fig. 16-12, and R11.12.6.2):

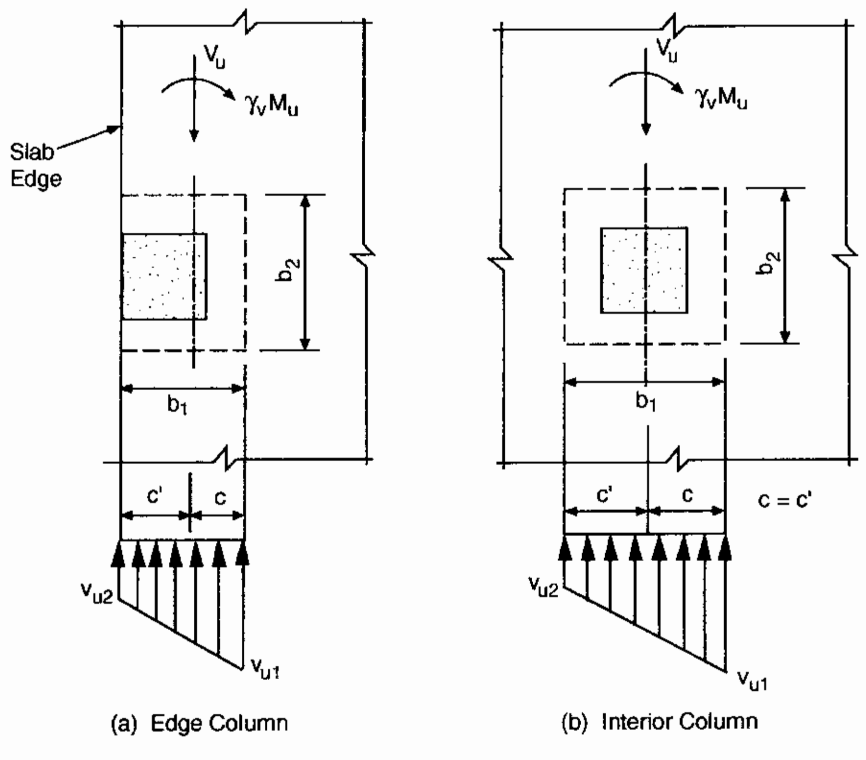


Fig. 16-12 Shear Stress Distribution due to Moment-Shear Transfer at Slab-Column Connection

$$v_{u1} = \frac{V_u}{A_c} + \frac{\gamma_v M_u c}{J}$$ *Eq. (1)*

$$v_{u2} = \frac{V_u}{A_c} - \frac{\gamma_v M_u c'}{J}$$ *Eq. (2)*

where: A_c = area of concrete section resisting shear transfer, equal to the perimeter b_o multiplied by the effective depth d

J = property of critical section analogous to polar moment of inertia of segments forming area A_c.

c and c' = distances from centroidal axis of critical section to the perimeter of the critical section in the direction of analysis

Expressions for A_c,c,c',J/c, and J/c', are contained in Fig. 16-13 for rectangular columns and Fig. 16-14 for circular interior columns.

Where biaxial moment transfer occurs, research has shown that the method for evaluating shear stresses due to moment transfer between slabs and column in R.11.12.6.2 is still applicable (Ref. 16.8). There is no need to superimpose the shear stresses due to moments transfer in two directions.

The maximum shear stress v_{u1} computed from Eq. (1) shall not exceed ϕv_n, where ϕv_n is determined from the following (11.12.6.2):

a. For slabs without shear reinforcement: $\phi v_n = \phi v_c$, where ϕv_n is the minimum of:

$$\phi v_c = \phi\left(2 + \frac{4}{\beta}\right)\sqrt{f'_c}$$ *Eq. (11-33)*

$$\phi v_c = \phi\left(2 + \frac{\alpha_s d}{b_o}\right)\sqrt{f'_c}$$ *Eq. (11-34)*

$$\phi v_c = \phi 4\sqrt{f'_c}$$ *Eq. (11-35)*

b. For slabs with shear reinforcement other than shearheads, ϕv_n is computed from (11.12.3):

$$\phi v_n = \phi\left(2\sqrt{f'_c} + \frac{A_v f_y}{b_o s}\right) \le \phi 6\sqrt{f'_c}$$ *Eqs. (11-15), (11.12.3.1), and (11.12.3.2)*

where A_v is the total area of shear reinforcement provided on the column sides and b_o is the perimeter of the critical section located at d / 2 distance away from the column perimeter, as defined by 11.12.1.2 (a). Due to the variation in shear stresses, as illustrated in Fig. 16-12, the computed area of shear reinforcement, if required, may be different from one column side to the other. The required area of shear reinforcement due to shear stress v_{u1} at its respective column side is:

$$A_v = (v_{u1} - \phi v_c)\frac{(c + d)s}{\phi f_y}$$ *Eq. (3)*

where $(c+d)$ is an effective "beam" width and $v_c = 2\sqrt{f'_c}$. However, R11.12.3 recommends symmetrical placement if shear reinforcement on all column sides. Thus, with symmetrical shear reinforcement assumed on all sides of the column, the required area A_v may be computed from:

$$A_v = (v_{u1} - \phi v_c)\frac{b_o s}{\phi f_y} \qquad \textit{Eq. (4)}$$

where A_v is the total area of required shear reinforcement to be extended from the sides of the column, and b_o is the perimeter of the critical section located at d/2 from the column perimeter. With symmetrical reinforcement on all column sides, the reinforcement extending from the column sides with lesscomputed shear stress provides torsional resistance in the strip of slab perpendicular to the direction of analysis.

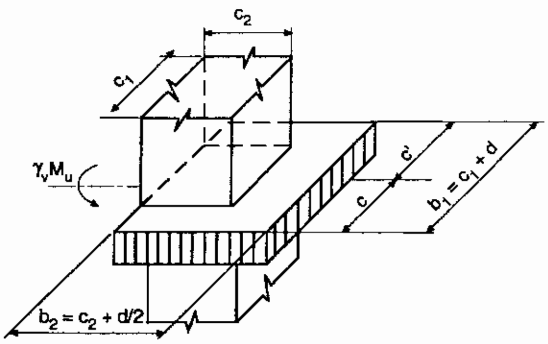
c. For slabs with shearheads as shear reinforcement, ϕv_n is computed from:

$$\phi v_n = \phi 4\sqrt{f'_c} \geq v_{u1} \qquad \textit{11.12.6.3}$$

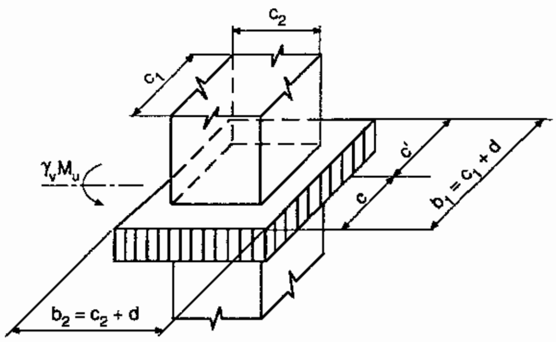
$$v_{u1} = \frac{V_u}{b_o d} + \frac{\gamma_v M_u c}{J} \leq \phi 4\sqrt{f'_c} \qquad \textit{Eq. (1)}$$

where b_o is the perimeter of the critical section defined in 11.12.4.7, c and J are section properties of the critical section located at d / 2 from the column perimeter (11.12.6.3), V_u is the direct shear force acting on the critical section defined in 11.12.4.7, and $\gamma_v M_u$ is the unbalanced moment transferred by eccentricity of shear acting about the centroid of the critical section defined in 11.12.1.2(a). Note that this seemingly inconsistent summation of shear stresses occurring at two different critical shear sections is conservative and justified by tests (see R11.12.6.3). At the critical section located d / 2 from the column perimeter, v_u shall not exceed $\phi 7\sqrt{f'_c}$ (11.12.4.8); see Fig. 16-5.

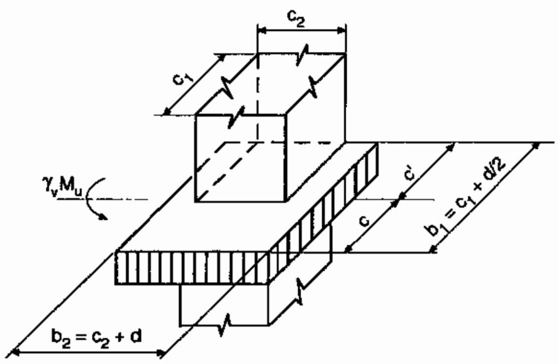
Case A: Edge Column (Bending parallel to edge)



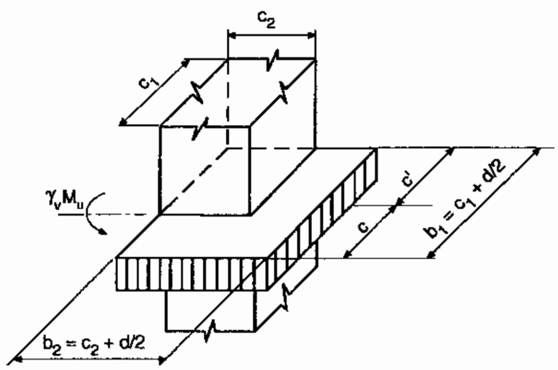
Case B: Interior Column



Case C: Edge Column (Bending perpendicular to edge)



Case D: Corner Column



Case	Area of critical section, A_C	Modulus of critical section		c	c'
		J/c	J/c'		
A	$(b_1+2b_2)d$	$\frac{b_1d(b_1+6b_2)+d^3}{6}$	$\frac{b_1d(b_1+6b_2)+d^3}{6}$	$\frac{b_1}{2}$	$\frac{b_1}{2}$
B	$2(b_1+b_2)d$	$\frac{b_1d(b_1+3b_2)+d^3}{3}$	$\frac{b_1d(b_1+3b_2)+d^3}{3}$	$\frac{b_1}{2}$	$\frac{b_1}{2}$
C	$(2b_1+b_2)d$	$\frac{2b_1^2d(b_1+2b_2)+d^3(2b_1+b_2)}{6b_1}$	$\frac{2b_1^2d(b_1+2b_2)+d^3(2b_1+b_2)}{6(b_1+b_2)}$	$\frac{b_1^2}{2b_1+b_2}$	$\frac{b_1(b_1+b_2)}{2b_1+b_2}$
D	$(b_1+b_2)d$	$\frac{b_1^2d(b_1+4b_2)+d^3(b_1+b_2)}{6b_1}$	$\frac{b_1^2d(b_1+4b_2)+d^3(b_1+b_2)}{6(b_1+2b_2)}$	$\frac{b_1^2}{2(b_1+b_2)}$	$\frac{b_1(b_1+2b_2)}{2(b_1+b_2)}$

Fig. 16-13 Section Properties for Shear Stress Computations – Rectangular Columns

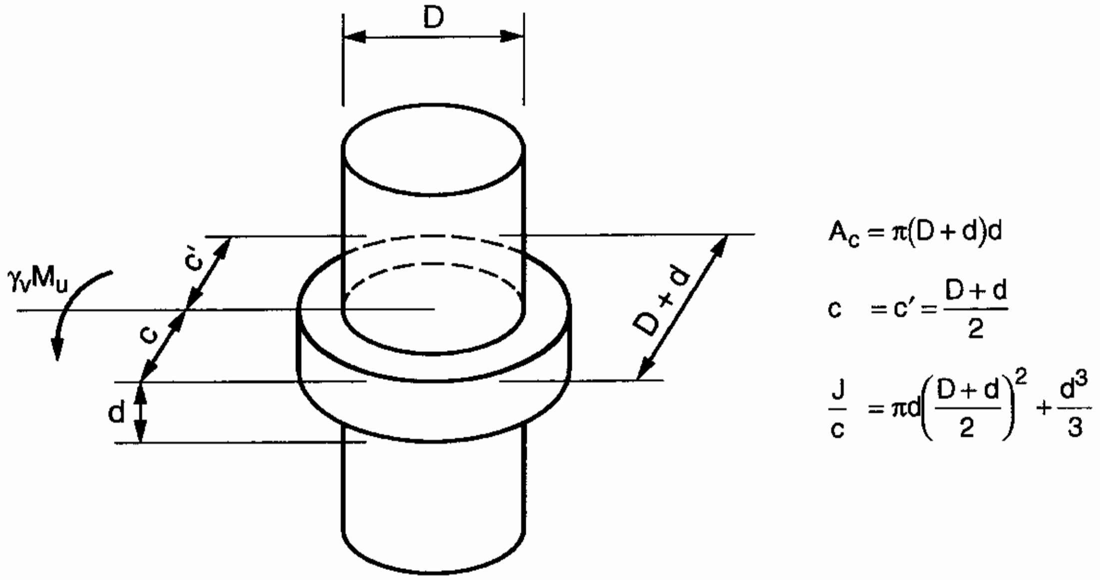


Fig. 16-14 Section Properties for Shear Stress Computations – Circular Interior Column

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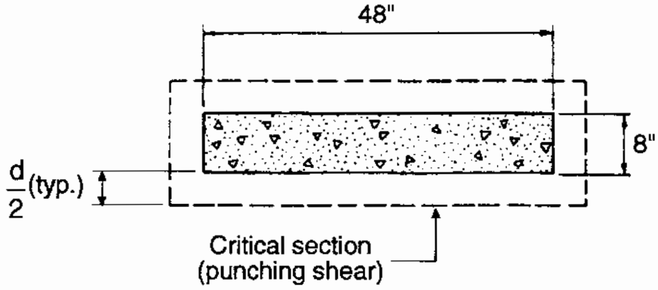
Example 16.1—Shear Strength of Slab at Column Support

Determine two-way action shear strength at an interior column support of a flat plate slab system for the following design conditions.

Column dimensions = 48 in. × 8 in.

Slab effective depth d = 6.5 in.

Specified concrete strength f'_c = 4,000 psi



Calculations and Discussion	Code Reference
1. Two-way action shear (punching shear) without shear reinforcement:	
$V_u \le \phi V_n$	*Eq. (11-1)*
$\le \phi V_c$	*11.12.2*
2. Effect of loaded area aspect ratio β_c:	
$\phi V_c = \phi\left(2 + \frac{4}{\beta}\right)\sqrt{f'_c}\, b_o d$	*Eq. (11-33)*
where $\beta = \frac{48}{8} = 6$	*11.12.2.1*
$b_o = 2\,(48 + 6.5 + 8 + 6.5) = 138$ in.	*11.12.1.2*
$\phi = 0.75$	*9.3.2.3*
$\phi V_c = 0.75 \times 138 \times 6.5/1{,}000 = 113.5$ kips	
3. Effect of perimeter area aspect ratio β_o:	
$\phi V_c = \phi\left(2 + \frac{\alpha_s}{\beta_o}\right)\sqrt{f'_c}\, b_o d$	*Eq. (11-34)*
where α_s = 40 for interior column support	*11.12.2.1*
$\beta_o = \frac{b_o}{d} = \frac{138}{6.5} = 21.2$	
$\phi V_c = 0.75 \times 138 \times 6.5/1{,}000 = 165.4$ kips	

Example 16.1 (cont'd)	Calculations and Discussion	Code Reference

4. Excluding effect of β and β_o:

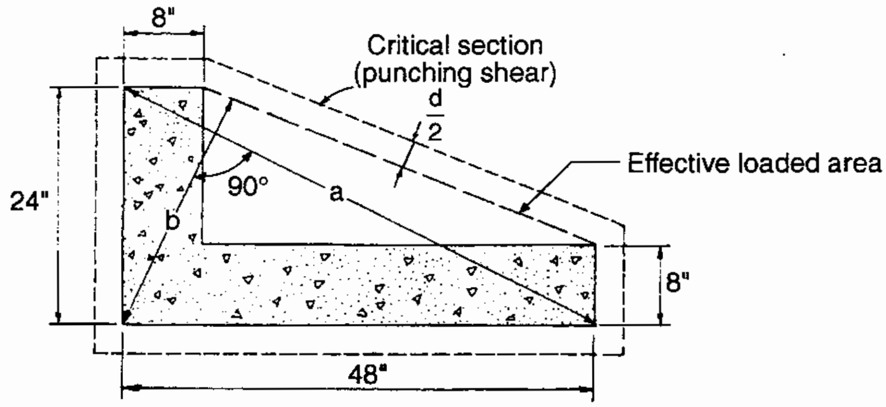
$$\phi V_c = \phi 4\sqrt{f_c'}\, b_o d$$ *Eq. (11-35*

$$= 0.75 \times 4 \times \sqrt{4{,}000} \times 138 \times 6.5/1{,}000 = 170.2 \text{ kips}$$

5. The shear strength ϕV_n is the smallest of the values computed above, i.e., $\phi V_n = 113.5$ kips.

Example 16.2—Shear Strength for Non-Rectangular Support

For the L-shaped interior column support shown, check punching shear strength for a factored shear force of V_u = 125 kips. Use f'_c = 4,000 psi. Effective slab depth = 5.5 in.



Calculations and Discussion	Code Reference

1. For shapes other than rectangular, R11.12.2.1 recommends that β be taken as the ratio of the longest overall dimension of the effective loaded area a to the largest overall dimension of the effective loaded area b, measured perpendicular to a: *R11.12.2.1*

$$\beta = \frac{a}{b} = \frac{54}{25} = 2.16$$

For the critical section shown, b_o = 141 in. *11.12.1.2*

Scaled dimensions of the drawings are used, and should be accurate enough

2. Two-way action shear (punching shear) without shear reinforcement:

$$V_u \le \phi V_n$$ *Eq. (11-1)*

$$\le \phi V_c$$ *11.12.2*

where the nominal shear strength V_c without shear reinforcement is the lesser of values given by Eqs. (11-33) and (11-34), but not greater than $4\sqrt{f'_c}\, b_o d$:

$$V_c = \left(2 + \frac{4}{\beta}\right)\sqrt{f'_c}\, b_o d$$ *Eq. (11-33)*

$$= \left(2+\frac{4}{2.16}\right)\sqrt{4{,}000} \times 141 \times 5.5/1{,}000 = 188.9 \text{ kips}$$

$$V_c = \left(2 + \frac{\alpha_s}{\beta_o}\right)\sqrt{f'_c}\, b_o d$$ *Eq. (11-34)*

where α_s = 40 for interior column support *11.12.2.1*

$$\beta_o = \frac{b_o}{d} = \frac{141}{5.5} = 25.6$$

$$V_c = \left(2 + \frac{40}{25.6}\right)\sqrt{4,000} \times 141 \times 5.5/1,000 = 174.7 \text{ kips}$$

$$V_c = 4\sqrt{f'_c}b_o d$$ *Eq. (11-35)*

$$= 4\sqrt{4,000} \times 141 \times 5.5/1,000 = 196.2 \text{ kips}$$

ϕV_c = 0.75 (174.7) = 131 kips

V_u = 125 kips < ϕV_c = 131 kips O.K.

Example 16.3—Shear Strength of Slab with Shear Reinforcement

Consider an interior panel of a flat plate slab system supported by a 12-in. square column. Panel size $\ell_1 = \ell_2 =$ 21 ft. Determine shear strength of slab at column support, and if not adequate, increase the shear strength by shear reinforcement. Overall slab thickness h = 7.5 in. (d = 6 in.).

$f'_c = 4{,}000$ psi

$f_y = 60{,}000$ psi (bar reinforcement)

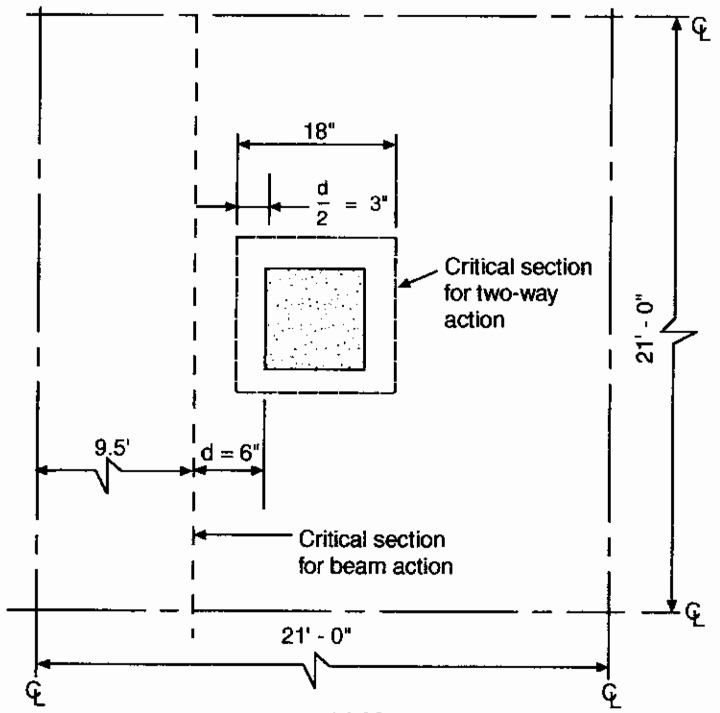
$f_y = 36{,}000$ psi (structural steel)

Superimposed factored load = 160 psf

Column strip negative moment M_u = 175 ft-kips

Calculations and Discussion	Code Reference
1. Wide-beam action shear and two-way action shear (punching shear) without shear reinforcement:	*11.12.2*
$V_u \leq \phi V_n$	*Eq. (11-1)*
$\leq \phi V_c$	*11.12.2*

a. Since there are no shear forces at the center lines of adjacent panels, tributary areas and critical sections for slab shear are as shown below.



Example 16.3 (cont'd)	Calculations and Discussion	Code Reference

For 7.5-in. slab, factored dead load $q_{Du} = 1.2 \times \frac{7.5}{12} \times 150 = 113$ psf *9.2.1*

$q_u = 113 + 160 = 273$ psf

a. Wide-Beam Action Shear.

Investigation of wide-beam action shear strength is made at the critical section at a distance d from face of column support. *11.1.3.1*

$V_u = 0.273\,(9.5 \times 21) = 54.5$ kips

$V_c = 2\sqrt{f'_c}b_w d = 2\sqrt{4{,}000}\,(21 \times 21) \times 6/1{,}000 = 191.3$ kips *Eq. (11-3)*

$\phi = 0.75$ *9.3.2.3*

$\phi V_c = 0.75\,(191.3) = 143.5 \text{ kips} > V_u = 54.5$ kips O.K.

Wide-beam action will rarely control the shear strength of two-way slab systems.

b. Two-Way Action Shear.

Investigation of two-way action shear strength is made at the critical section b_o located at d/2 from the column perimeter. Total factored shear force to be transferred from slab to column: *11.12.1.2(a)*

$V_u = 0.273\,(21^2 - 1.5^2) = 119.8$ kips

Shear strength V_c without shear reinforcement: *11.12.2.1*

$b_o = 4\,(18) = 72$ in. *11.12.1.2(a)*

$$\beta = \frac{12}{12} = 1.0 < 2$$

$$\beta_o = \frac{b_o}{d} = \frac{72}{6} = 12 < 20$$

$$V_c = 4\sqrt{f'_c}b_o d = 4\sqrt{4{,}000} \times 72 \times 6/1{,}000 = 109.3 \text{ kips}$$

$\phi = 0.75$ *9.3.2.3*

$\phi V_c = 0.75\,(109.3) = 82 \text{ kips} < V_u = 119.8$ kips N.G.

Shear strength of slab is not adequate to transfer the factored shear force $V_u = 119.8$ kips from slab to column support. Shear strength may be increased by:

i. increasing concrete strength f'_c
ii. increasing slab thickness at column support, i.e., using a drop panel
iii. providing shear reinforcement (bars, wires, or steel I- or channel-shapes)

The following parts of the example will address all methods to increase shear strength.

2. Increase shear strength by increasing strength of slab concrete:

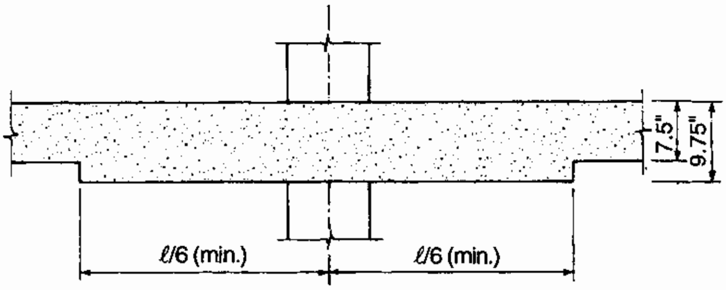
   $V_u \leq \phi V_n$ *Eq. (11-1)*

   $119{,}800 \leq 0.75\left(4\sqrt{f'_c} \times 72 \times 6\right)$

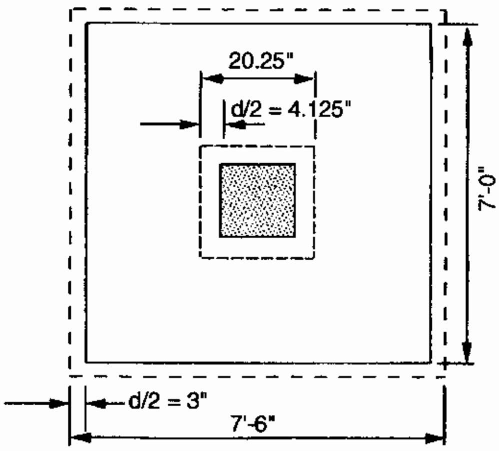
   Solving, $f'_c = 8{,}545$ psi

3. Increase shear strength by increasing slab thickness at column support with drop panel:

   Provide drop panel in accordance with 13.2.5 (see Fig. 18-18). Minimum overall slab thickness at drop panel = 1.25 (7.5) = 9.375-in. Try a 9.75 in. slab thickness (2.25-in. projection below slab*; d ≈ 8.25 in.). Minimum distance from centerline of column to edge of drop panel = 21/6 = 3.5 ft. Try 7 × 7 ft drop panel.



Drop Panel Section



   a. Investigate shear strength at critical section b_o located at d/2 from column perimeter.

      Total factored shear force to be transferred —

* See Chapter 9 (Design Considerations for Economical Formwork) in Ref. 16.6.

For 2.25-in. drop panel projection, $q_{D_u} = 1.2 \times \frac{2.25}{12} \times 150 = 34$ psf

$V_u = 0.273\,(21^2 - 1.69^2) + 0.034\,(7^2 - 1.69^2) = 119.6 + 1.6 = 121.2$ kips

$b_o = 4\,(12 + 8.25) = 81$ in. *11.12.1.2(a)*

$\beta = 1.0 < 2$

$\beta_o = \frac{b_o}{d} = \frac{81}{8.25} = 9.8 < 20$

$\phi V_c = \phi 4\sqrt{f'_c}\, b_o d$ *Eq. (11-35)*

$= 0.75 \times 4\sqrt{4{,}000} \times 81 \times 8.25 > V_u = 121.2$ kips O.K.

b. Investigate shear strength at critical section b_o located at d/2 from edge of drop panel.

Total factored shear force to be transferred —

$V_u = 0.273\,(21^2 - 7.5^2) = 105.0$ kips

$b_o = 4\,(84 + 6) = 360$ in. *11.12.1.2(b)*

$\beta = \frac{84}{84} = 1.0 < 2$ *Eq. (11-35)*

$\beta_o = \frac{b_o}{d} = \frac{360}{6} = 60 > 20$

$\phi V_c = \phi\left(2 + \frac{\alpha_s}{\beta_o}\right)\sqrt{f'_c}\, b_o d = \phi\left(2 + \frac{40}{60}\right)\sqrt{f'_c}\, b_o d = \phi 2.67\sqrt{f'_c}\, b_o d$ *Eq. (11-36)*

$= 0.75 \times 2.67\sqrt{4{,}000} \times 360 \times 6 / 1{,}000 = 273.2$ kips $> V_u = 105.0$ kips O.K.

Note the significant decrease in potential shear strength at edge of drop panel due to large β_o.

A 7 × 7 ft drop panel with a 2.25-in. projection below the slab will provide adequate shear strength for the superimposed factored loads of 160 psf.

Example 16.3 (cont'd)	Calculations and Discussion	Code Reference

4. Increase shear strength by bar reinforcement (see Figs. R11.12.3(a) and 16-5):

a. Check effective depth d — *11.12.3*

Assuming No. 3 stirrups ($d_b = 0.375$ in.),

$$d = 6 \text{ in.} \geq \begin{cases} 6 \text{ in. O.K.} \\ 16 \times 0.375 = 6 \text{ in. O.K.} \end{cases}$$

b. Check maximum shear strength permitted with bars. — *11.12.3.2*

$V_u \leq \phi V_n$ — *Eq. (11-1)*

$$\phi V_n = \phi\left(6\sqrt{f'_c}b_o d\right) = 0.75\left(6\sqrt{4{,}000} \times 72 \times 6\right)/1{,}000 = 123.0 \text{ kips}$$

$$V_u = 0.273\,(21^2 - 1.5^2) = 119.8 \text{ kips} < \phi V_n = 123.0 \text{ kips O.K.}$$

c. Determine shear strength provided by concrete with bar shear reinforcement. — *11.12.3.1*

$$V_c = 2\sqrt{f'_c}b_o d = 2\sqrt{4{,}000} \times 72 \times 6/1{,}000 = 54.6 \text{ kips}$$

$$\phi V_c = 0.75\,(54.6) = 41.0 \text{ kips}$$

d. Design shear reinforcement in accordance with 11.5.

Required area of shear reinforcement A_v is computed by

$$A_v = \frac{(V_u - \phi V_c)\,s}{\phi f_y d}$$

Assumes = 3 in. (maximum spacing permitted = d/2) — *11.5.4.1*

$$A_v = \frac{(119.8 - 41.0) \times 3}{0.75 \times 6.0 \times 6} = 0.88 \text{ in.}^2$$

where A_v is total area of shear reinforcement required on the four sides of the column (see Fig. 16-5).

$$A_v \text{ (per side)} = \frac{0.88}{4} = 0.22 \text{ in.}^2$$

e. Determine distance from sides of column where stirrups may be terminated (see Fig. 16-5).

$V_u \leq \phi V_c$ *Eq. (11-1)*

$\leq \phi 2\sqrt{f'_c}\, b_o d$

For square column (see sketch below),

$b_o = 4\left(12 + a\sqrt{2}\right)$

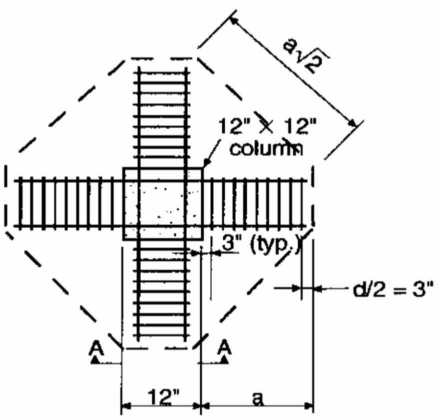
$119{,}800 \leq 0.75 \times 2\sqrt{4000} \times 4\left(12 + a\sqrt{2}\right) \times 6$

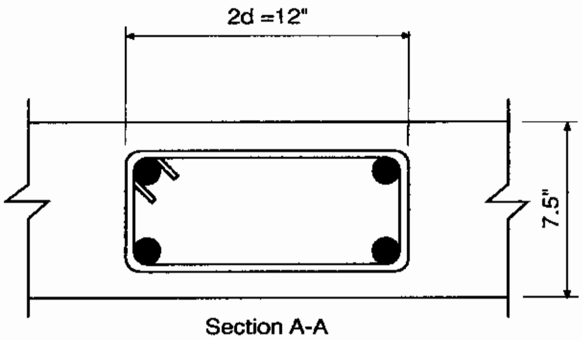
Solving, a = 28.7 in.

Note that the above is a conservative estimate, since V_u at the perimeter of the critical section shown below is considerably lower than 119.8 kips.

Stirrups may be terminated at d/2 = 3 in. inside the critical perimeter b_o.

Use 9-No. 3 closed stirrups @ 3 in. spacing (A_v = 0.22 in.^{2}) along each column line as shown below.





Section A-A

5. Increase shear strength by steel I shapes (shearheads): *11.12.4*

a. Check maximum shear strength permitted with steel shapes (see Fig. 18-8). *11.12.4.8*

$V_u = 0.273\,(21^2 - 1.5^2) = 119.8$ kips

$V_u \leq \phi V_n$ *Eq. (11-1)*

$\phi V_n = \phi\left(7\sqrt{f'_c}\, b_o d\right)$ *11.12.4.8*

$\leq 0.75\left(7\sqrt{4{,}000} \times 72 \times 6\right)/1{,}000 = 143.4 \text{ kips} > V_u = 119.8 \text{ kips}$ O.K.

b. Determine minimum required perimeter b_o of a critical section at shearhead ends with shear strength limited to $V_n = 4\sqrt{f'_c}\, b_o d$ (see Fig. 16-6 (b)).

$V_u \le \phi V_n$ *Eq. (11-1)*

$119{,}800 \le 0.75\left(4\sqrt{4{,}000} \times b_o \times 6\right)$

Solving, b_o = 105.2 in. *11.12.4.7*

c. Determine required length of shearhead arm ℓ_v to satisfy b_o = 105.2 in. at 0.75 $(\ell_v - c_1/2)$. *11.12.4.7*

$b_o \approx 4\sqrt{2}\left[\frac{c_1}{2} + \frac{3}{4}\left(\ell_v - \frac{c_1}{2}\right)\right]$ (see Fig. 16-6 (b))

With b_o = 105.2 in. and c_1 = 12 in., solving, ℓ_v = 22.8 in.

Note that the above is a conservative estimate, since V_u at the perimeter of the critical section considered is considerably lower than 119.8 kips.

d. To ensure that premature flexural failure of shearhead does not occur before shear strength of slab is reached, determine required plastic moment strength M_p of each shearhead arm.

$$\phi M_p = \frac{V_u}{2n}\left[h_v + \alpha_v\left(\ell_v - \frac{c_1}{2}\right)\right]$$ *Eq. (11-37)*

For a four (identical) arm shearhead, n = 4; assuming h_v = 4 in. and α_v = 0.25: *11.12.4.5*

$$\phi M_p = \frac{119.8}{2(4)}\left[4 + 0.25\left(23.6 - \frac{12}{2}\right)\right] = 125.8 \text{ in.-kips}$$

$\phi = 0.9$ (tension-controlled member) *9.3.2.1*

Required $M_p = \frac{125.8}{0.9} = 139.8$ in.-kips

Try W4 × 13 (plastic modulus Z_x = 6.28 in.^{3}) A36 steel shearhead

$M_p = Z_x f_y$ = 6.28 (36) = 226.1 in.-kips > 139.8 in.-kips O.K.

e. Check depth limitation of W4 × 13 shearhead. *11.12.4.2*

$70t_w$ = 70 (0.280) = 19.6 in. > h_v = 4.16 in. O.K.

f. Determine location of compression flange of steel shape with respect to compression surface of slab, assuming 3/4-in. cover and 2 layers of No. 5 bars. *11.12.4.4*

0.3d = 0.3 (6) = 1.8 in. < 0.75 + 2 (0.625) = 2 in. N.G.

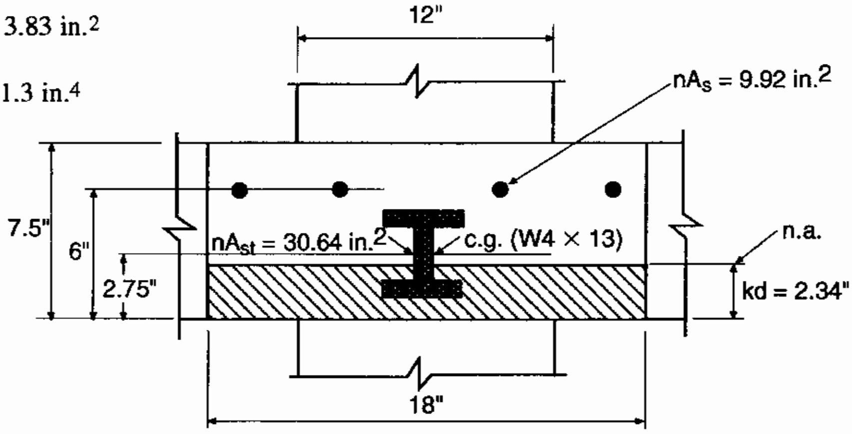
Therefore, both layers of the No. 5 bars in the bottom of the slab must be cut.

g. Determine relative stiffness ratio α_v.

For the W4 × 13 shape:

$A_{st} = 3.83 \text{ in.}^2$

$I_s = 11.3 \text{ in.}^4$



A_s provided for M_u = 175 ft-kips is No. 5 @ 5 in.

c.g. of W4 × 13 from compression face = 0.75 + 2 = 2.75 in.

Effective slab width = $c_2 + d$ = 12 + 6 = 18 in.

Transformed section properties:

For f'_c = 4,000 psi, use $\frac{E_s}{E_c} = \frac{29{,}000}{3605} = 8$

Steel transformed to equivalent concrete:

$$\frac{E_s}{E_c} A_s = 8\,(4 \times 0.31) = 9.92 \text{ in.}^2$$

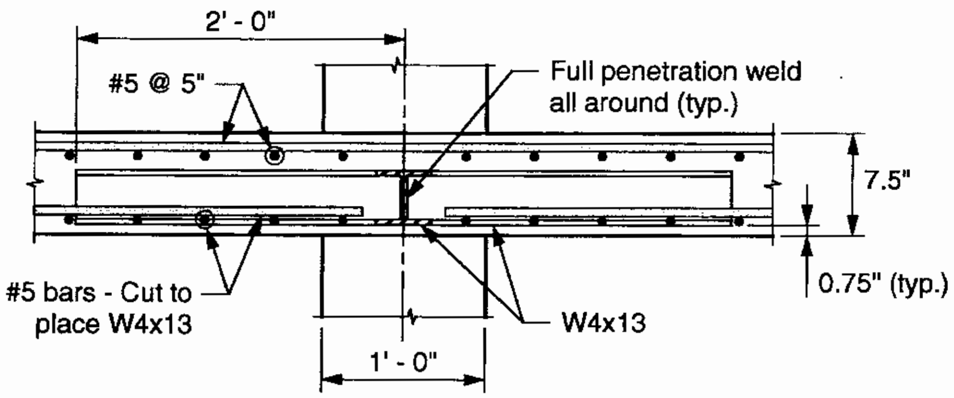
$$\frac{E_s}{E_c} A_{st} = 8\,(3.83) = 30.64 \text{ in.}^2$$

Neutral axis of composite cracked slab section may be obtained by equating the static moments of the transformed areas.

$$\frac{18\,(kd)^2}{2} = 30.64\,(2.75 - kd) + 9.92\,(6 - kd)$$

where kd is the depth of the neutral axis for the transformed area

Solving, kd = 2.34 in.



Final Details of Shearhead Reinforcement

$$\text{Composite I} = \frac{18\,(2.34)^3}{3} + \frac{E_s}{E_c}\,(I_s \text{ steel shape}) + 9.92\,(3.66)^2 + 30.64\,(0.41)^2$$

$$= 76.9 + 8\,(11.3) + 132.9 + 5.2 = 305.4 \text{ in.}^4$$

$$\alpha_v = \frac{E_s/E_c I_s}{I_{composite}} = \frac{8 \times 11.3}{305.4} = 0.30 > 0.15 \qquad \text{O.K.}$$

Therefore, W4 × 13 section satisfies all code requirements for shearhead reinforcement.

h. Determine contribution of shearhead to negative moment strength of column strip. *11.12.4.9*

$$M_v = \frac{\phi \alpha_v V_u}{2n}\left(\ell_v - \frac{c_1}{2}\right)$$ *Eq. (11-38)*

$$= \frac{0.9 \times 0.30 \times 119.8}{2 \times 4}(25 - 6) = 76.8 \text{ in.-kips} = 6.4 \text{ ft-kips}$$

However, M_v must not exceed either M_p = 139.8 in.-kips or 0.3 × 175 × 12 = 630 in.-kips, or the change in column strip moment over the length ℓ_v. For this design, approximately 4% of the column strip negative moment may be considered resisted by the shearhead reinforcement.

Example 16.4—Shear Strength of Slab with Transfer of Moment

Consider an exterior (edge) panel of a flat plate slab system supported by a 16-in. square column. Determine shear strength for transfer of direct shear and moment between slab and column support. Overall slab thickness h = 7.25 in. (d ≈ 6.0 in.). Assume that the Direct Design Method is used for analysis of the slab. Consider two loading conditions:

1. Total factored shear force V_u = 30 kips

   Total factored static moment M_o in the end span = 96 ft-kips

2. V_u = 60 kips

   M_o = 170 ft-kips

   f'_c = 4,000 psi

   f_y = 60,000 psi

Calculations and Discussion	Code Reference

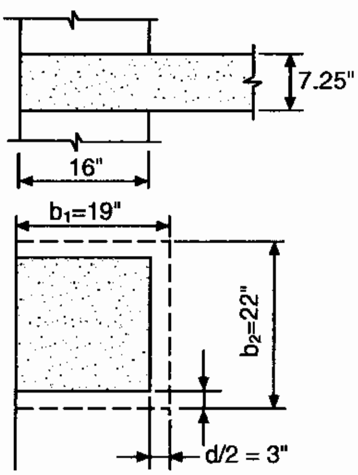
1. Section properties for shear stress computations:

   Referring to Fig. 16-13, edge column bending perpendicular to edge (Case C),

$$b_1 = c_1 + \frac{d}{2} = 16 + \frac{6}{2} = 19.0 \text{ in.}$$

$$b_2 = c_2 + d = 16 + 6 = 22.0 \text{ in.}$$

$$b_o = 2\,(19.0) + 22 = 60.0 \text{ in.}$$



$$c = \frac{b_1^2}{2b_1 + b_2}$$

$$= \frac{19.0^2}{(2\times 19.0) + 22.0} = 6.02 \text{ in.}$$

$$A_c = (2b_1 + b_2)\,d = 360 \text{ in.}^2$$

$$\frac{J}{c} = \frac{\left[2b_1^2 d(b_1 + 2b_2) + d^3(2b_1 + b_2)\right]}{6b_1} = 2{,}508 \text{ in.}^3$$

$$c' = b_1 - c = 19 - 6.02 = 12.98 \text{ in.}$$

$$\frac{J}{c'} = \left(\frac{J}{c}\right)\left(\frac{c}{c'}\right) = 2{,}508\left(\frac{6.02}{12.98}\right) = 1{,}163 \text{ in.}^3$$

2. Loading condition (1), $V_u = 30$ kips, $M_o = 96$ ft-kips:

a. Portion of unbalanced moment to be transferred by eccentricity of shear. *11.12.6.1*

$\gamma_v = 1 - \gamma_f$ *Eq. (11-39)*

For unbalanced moments about an axis parallel to the edge at exterior supports, the value of γ_f can be taken equal to 1.0 provided that $V_u \leq 0.75\phi V_c$. *13.5.3.3*

$V_c = 4\sqrt{f'_c}\, b_o d$ *Eq. (11-35)*

$= 4\sqrt{4{,}000} \times 60 \times 6.0/1{,}000 = 91.1$ kips

$\phi = 0.75$ *9.3.2.3*

$0.75\phi V_c = 0.75 \times 0.75 \times 91.1 = 51.2 \text{ kips} > V_u = 30 \text{ kips}$

Therefore, all of the unbalanced moment at the support may be considered transferred by flexure (i.e., $\gamma_f = 1.0$ and $\gamma_v = 0$). Note that γ_f can be taken as 1.0 provided that ρ within the effective slab width $3h + c_2 = 21.75 + 16 = 37.75$ in. is not greater than $0.375\rho_b$.

b. Check shear strength of slab without shear reinforcement.

Combined shear stress along inside face of critical transfer section.

$$v_{u1} = \frac{V_u}{A_c} + \frac{\gamma_v M_u c}{J} = \frac{30{,}000}{360} + 0 = 83.3 \text{ psi}$$

Permissible shear stress:

$$\phi v_n = \phi 4\sqrt{f'_c} = 0.75\left(4\sqrt{4{,}000}\right) = 189.7 \text{ psi} > v_{u1} = 83.3 \text{ psi O.K.}$$

Slab shear strength is adequate for the required shear and moment transfer between slab and column.

Design for the portion of unbalanced moment transferred by flexure $\gamma_f M_u$ must also be considered. See Example 19.1 when using the Direct Design Method. See Example 20.1 for the Equivalent Frame Method. *13.5.3.2*

For the Direct Design Method, $\gamma_f M_u = 1.0 \times (0.26\, M_o) = 25$ ft-kips to be transferred over the effective width of 37.75 in., provided that ρ within the 37.75-in. width $\leq 0.375\,\rho_b$. *13.6.3.3* *13.5.3.3*

$$= 0.75\left[2\sqrt{4,000} + \frac{(3\times 0.22)\times 60,000}{60\times 3.0}\right]$$

$$= 0.75\,(126.5 + 220.0) = 259.9 \text{ psi} > v_{u1} = 259.4 \text{ psi} \quad \text{O.K.}$$

e. Determine distance from sides of column where stirrups may be terminated.

$V_u \le \phi V_c$ *11.12.3.1*

$\phi V_c = \phi 2\sqrt{f'_c}\, b_o d$

where $b_o = 2a\sqrt{2} + (3 \times 16)$

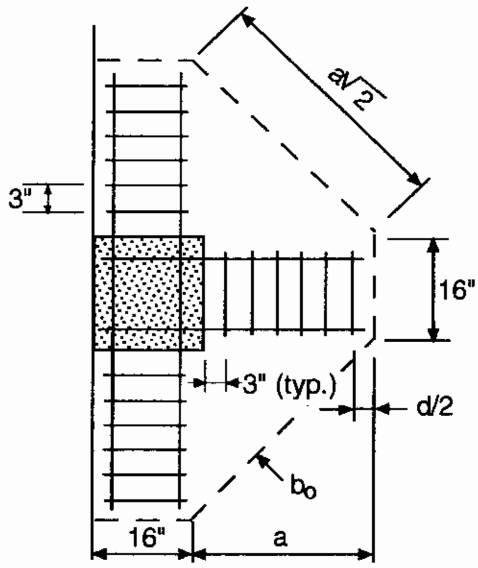
$60,000 \le 0.75\times 2\sqrt{4,000}\left(2a\sqrt{2} + 48\right)6.0$

Solving, a = 20.3 in.

Note that the above is a conservative estimate, since V_u at the perimeter of the critical section considered is considerably lower than 60 kips.

No. of stirrups required = (20.3 - d/2)/3.0 = 5.8
(Stirrups may be terminated at d/2 = 3.0 in. inside perimeter b_o)

Use 6-No. 3 closed stirrups @ 3.0 in. spacing along the three sides of the column.
Use similar stirrup detail as for Example 16.3.



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17

Strut-and-Tie Models

GENERAL

The strut-and-tie model is essentially a truss analogy. It is based on the fact that concrete is strong in compression, and that steel is strong in tension. Truss members that are in compression are made up of concrete, while truss members that are in tension consist of steel reinforcement.

Appendix A, Strut-and-Tie Models, was introduced in ACI 318-02. It provides a design approach, applicable to an array of design problems that do not have an explicit design solution in the body of the code. This method requires the designer to consciously select a realistic load path within the structural member in the form of an idealized truss. Rational detailing of the truss elements and compliance with equilibrium assures the safe transfer of loads to the supports or to other regions designed by conventional procedures. While solutions provided with this powerful design and analysis tool are not unique, they represent a conservative lower bound approach. As opposed to some of the prescriptive formulations in the body of ACI 318, the very visual, rational strut-and-tie model of Appendix A gives insight into detailing needs of irregular regions of concrete structures and promotes ductility.

The design methodology presented in Appendix A is largely based on the seminal articles on the subject by Schlaich et al.[17.1], Collins and Mitchell[17.2], and Marti[17.3]. Since publication of these papers, the strut-and-tie method has received increased attention by researchers and textbook writers (Collins and Mitchell[17.4], MacGregor and Wight[17.5]). MacGregor described the background of provisions incorporated in Appendix A in ACI Special Publication SP-208[17.6]. The present form of Appendix A does not include explicit serviceability provisions (such as deflection control).

A.1 DEFINITIONS

The strut-and-tie design procedure calls for the distinction of two types of zones in a concrete component depending on the characteristics of stress fields at each location. Thus, structural members are divided into B-regions and D-regions.

B-regions represent portions of a member in which the "plane section" assumptions of the classical beam theory can be applied with a sectional design approach.

D-regions are all the zones outside the B-regions where cross-sectional planes do not remain plane upon loading. D-regions are typically assumed at portions of a member where **discontinuities** (or disturbances) of stress distribution occur due to concentrated forces (loads or reactions) or abrupt changes of geometry. Based on St. Venant's Principle, the normal stresses (due to axial load and bending) approach quasi-linear distribution at a distance approximately equal to the larger of the overall height (h) and width of the member, away from the location of the concentrated force or geometric irregularity. Figure 17-1 illustrates typical discontinuities, D-Regions (cross-hatched areas), and B-Regions.

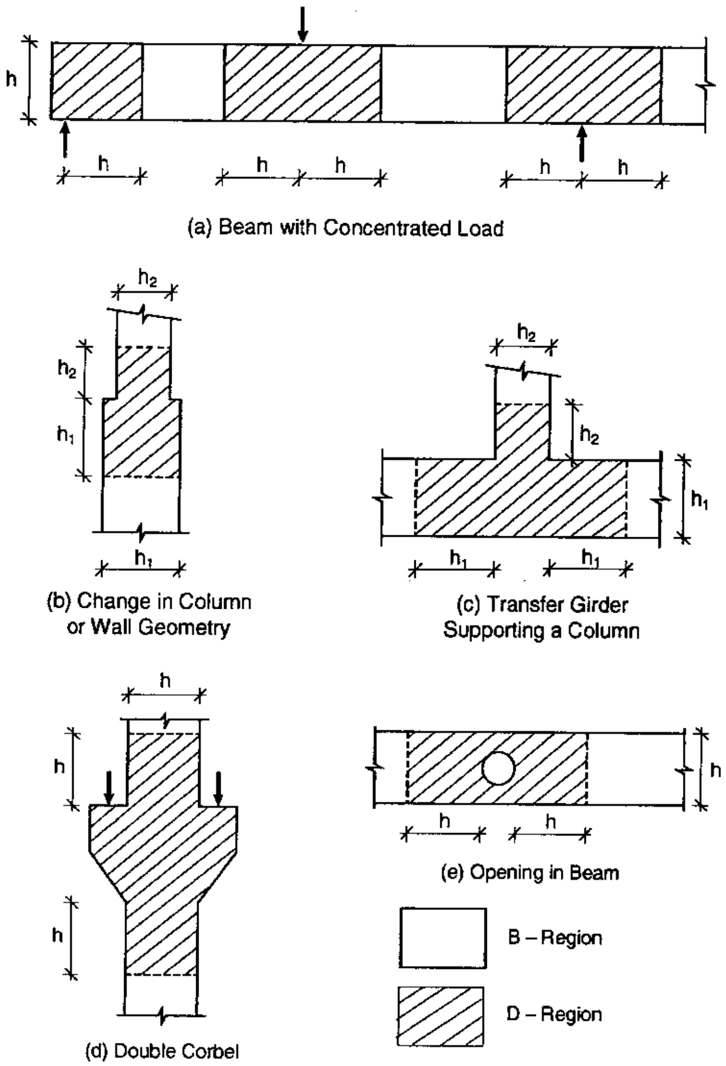

Figure 17-1 Load and Geometric Discontinuities

While B-regions can be designed with the traditional methods (ACI 318 Chapters 10 and 11), the **strut-and-tie model** was primarily introduced to facilitate the design of D-regions, and can be extended to the B-regions as well. The strut-and-tie model depicts the D-region of the structural member with a truss system consisting of compression <u>struts</u> and tension <u>ties</u> connected at <u>nodes</u> as shown in Fig. 17-2. This truss system is designed to transfer the factored loads to the supports or to adjacent B-regions. At the same time, forces in the truss members should maintain equilibrium with the applied loads and reactions.

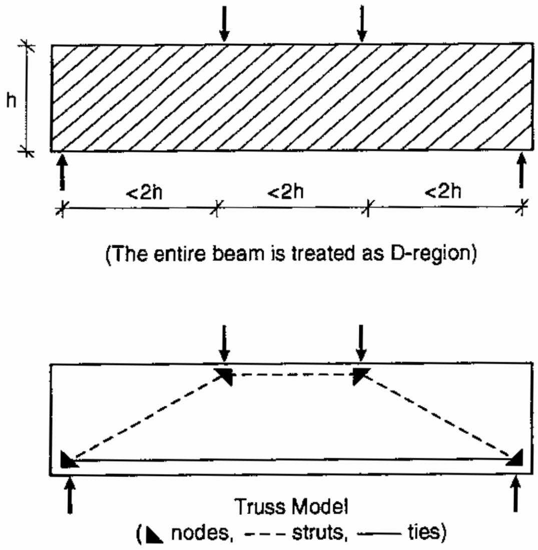

Figure 17-2 Strut-and-Tie Model

Struts are the compression elements of the strut-and-tie model representing the resultants of a compression field. Both parallel and fan shaped compression fields can be modeled by their resultant compression struts as shown in Fig. 17-3.

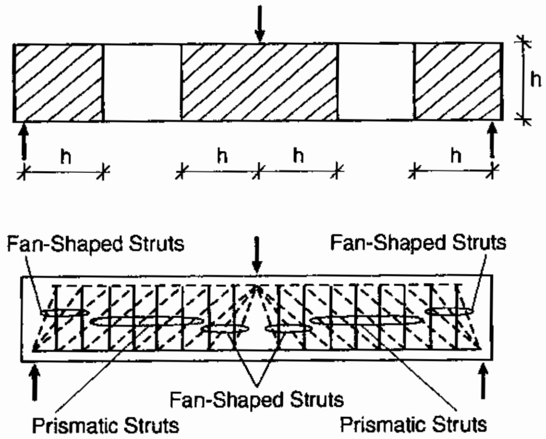

Figure 17-3 Prismatic and Fan-Shaped Struts

Typically compression struts would take a bottle-shape wherever the strut can spread laterally at mid-length. As a design simplification, prismatic compression members commonly idealize struts, however, other shapes are also possible. If the effective concrete compressive strength (f_{ce}) is different at the opposite ends of a strut, a linearly tapering compression member is suggested. This condition may occur, if at the two ends of the strut the nodal zones have different strengths or different bearing lengths. Should the compression stress be high in the strut, reinforcement may be necessary to prevent splitting due to transverse tension. (The splitting crack that develops in a cylinder supported on edge, and loaded in compression is a good example of the internal lateral spread of the compressive stress trajectories).

Ties consist of conventional deformed steel, or prestressing steel, or both, plus a portion of the surrounding concrete that is concentric with the axis of the tie. The surrounding concrete is not considered to resist axial force in the model. However, it reduces the elongation of the tie (tension stiffening), in particular, under service loads. It also defines the zone in which the forces in the struts and ties are to be anchored.

Nodes are the intersection points of the axes of the struts, ties and concentrated forces, representing the joints of a strut-and-tie model. To maintain equilibrium, at least three forces should act on a given node of the model. Nodes are classified depending on the sign of the forces acting upon them (e.g., a C-C-C node resists three compression forces, a C-T-T node resists one compression forces and two tensile forces, etc.) as shown in Fig. 17-4

Figure 17-4 Classification of Nodes

A nodal zone is the volume of concrete that is assumed to transfer strut-and tie forces through the node. The early strut-and-tie models used hydrostatic nodal zones, which were lately superseded by extended nodal zones.

The faces of a **hydrostatic nodal zone** are perpendicular to the axes of the struts and ties acting on the node, as depicted in Fig. 17-5. The term hydrostatic refers to the fact that the in-plane stresses are the same in all directions. (Note that in a true hydrostatic stress state the out-of plane stresses should be also equal). Assuming identical stresses on all faces of a C-C-C nodal zone with three struts implies that the ratios of the lengths of the sides of the nodal zones ($w_{n1} : w_{n2} : w_{n3}$) are proportional to the magnitude of the strut forces ($C_1 : C_2 : C_3$).

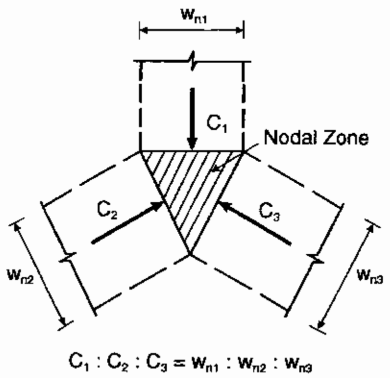

Figure 17-5 Hydrostatic Nodal Zone

The extended nodal zone is a portion of a member bounded by the intersection of the effective strut width, w_s, and the effective tie width, w_t. This is illustrated in Fig. 17-6.

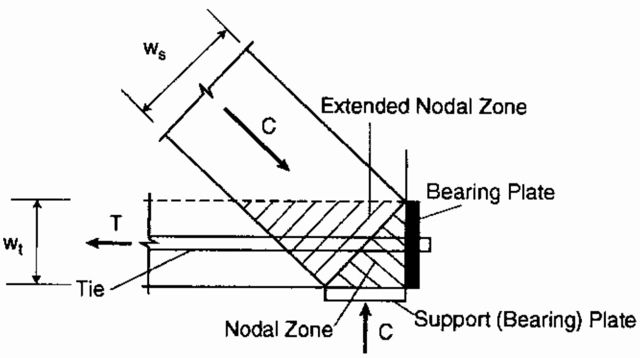

Figure 17-6 Extended Nodal Zone

A.2 STRUT-AND-TIE MODEL DESIGN PROCEDURE

A design with the strut-and-tie model typically involves the following steps:

1. Define and isolate D-regions.
2. Compute resultant forces on each D-region boundary.
3. Devise a truss model to transfer the resultant forces across the D-region. The axes of the struts and ties, respectively, are oriented to approximately coincide with the axes of the compression and tension stress fields.
4. Calculate forces in the truss members.
5. Determine the effective widths of the struts and nodal zones considering the forces from the previous steps and the effective concrete strengths (defined in A.3.2 and A.5.2). Strength checks are based on

   $$\phi F_n \geq F_u \qquad \textit{Eq. (A-1)}$$

   where F_u is the largest factored force obtained from the applicable load combinations, F_n is the nominal strength of the strut, tie, or node, and the ϕ factor is listed in 9.3.2.6 as 0.75 for ties, strut, nodal zones and bearing areas of strut-and-tie models.
6. Provide reinforcement for the ties considering the steel strengths defined in A.4.1. The reinforcement must be detailed to provide proper anchorage in the nodal zones

In addition to the strength limit states, represented by the strut-and-tie model, structural members should be checked for serviceability requirements. Traditional elastic analysis can be used for deflection checks. Crack control can be verified using provisions of 10.6.4, assuming that the tie is encased in a prism of concrete corresponding to the area of tie (RA.4.2).

There are usually several strut-and-tie models that can be devised for a given structural member and loading condition. Models that satisfy the serviceability requirements the best, have struts and ties that follow the compressive and tensile stress trajectories, respectively. Certain construction rules of strut-and-tie models, e.g., "the angle, θ, between the axes of any strut and any tie entering a single node shall not be taken as less than 25 degree (A.2.5) are imposed to mitigate potential cracking problems and to avoid incompatibilities due to shortening of the struts and lengthening of the ties in almost the same direction.

A.3 STRENGTH OF STRUTS

The nominal compressive strength of a strut without longitudinal reinforcement shall be taken as

$$F_{ns} = f_{ce} A_{cs} \qquad \textit{Eq. (A-2)}$$

to be calculated at the weaker end of the compression member. A_{cs} is the cross-sectional area at the end of the strut. In typical two-dimensional members, the width of the strut (w_s) can be taken as the width of the member. The effective compressive strength of the concrete (f_{ce}) for this purpose shall be taken as the lesser of the concrete strengths at the two sides of the nodal zone/strut interface. Section A.3.2 specifies the calculation of f_{ce} for the strut (detailed below), while A.5.2 provides for the same in the nodal zone (discussed later).

The effective compressive strength of the concrete in a strut is calculated, similarly to basic strength equations, as:

$$f_{ce} = 0.85\beta_s f_c' \qquad \textit{Eq. (A-3)}$$

The β_s factor accounts for the effect of cracking and possible presence of transverse reinforcement. The strength of the concrete in a strut can be computed with $\beta_s = 1.0$ for struts that have uniform cross sectional area over their length. This is quasi-equivalent to the rectangular stress block in the compression zone of a beam or column. For bottle-shaped struts (Fig. 17-7) with reinforcement placed to resist the splitting forces (satisfying A3.3) $\beta_s = 0.75$ or without adequate confinement to resist splitting forces $\beta_s = 0.6\lambda$ (where λ is a correction factor (11.7.4.3) for lightweight concrete.)

For struts intersecting cracks in a tensile zone, β_s is reduced to 0.4. Examples include strut-and-tie models used to design the longitudinal and transverse reinforcement of the tension flanges of beams, box-girders and walls. For all other cases (e.g., in beam webs where struts are likely to be crossed by inclined cracks), the β_s factor can be conservatively taken as 0.6.

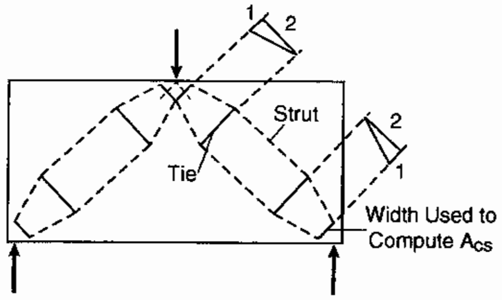

Figure 17-7 Bottle Shaped Compression Strut

Section A.3.3 addresses cases where transverse reinforcement is provided to cross the bottle-shaped struts. The compression forces in the strut may be assumed to spread at a slope 2:1. The rebars are intended to resist the transverse tensile forces resulting from the compression force spreading in the strut. They may be placed in one layer (when the γ angle between the rebar and the axis of the strut is at least 40 degree) or in two orthogonal layers.

To allow for $\beta_s = 0.75$, for concrete strength not exceeding 6000 psi, the reinforcement ratio needed to cross the strut is:

$$\Sigma \frac{A_{si}}{b_s s_i} \sin \gamma_i \geq 0.003 \qquad \textit{Eq. (A-4)}$$

where A_{si} is the total area of reinforcement at spacing s_i in a layer of reinforcement with bars at an angle γ_i to the axis of the strut (shown in Fig. 17-8), and b_s is the width of the strut. Often, this reinforcement ratio cannot be provided due to space limitations. In those cases $\beta_s = 0.6\lambda$ shall be used.

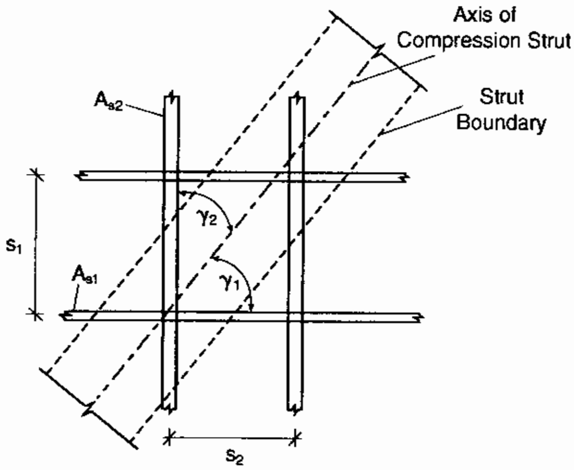

Figure 17-8 Layers of Reinforcement to Restrain Splitting Cracks of Struts

If substantiated by test and analyses, increased effective compressive strength of a strut due to confining reinforcement may be used (e.g., at anchorage zones of prestressing tendons). This topic is discussed in detail in Refs. 17.7 and 17.8.

Additional strength can be provided to the struts by including compression reinforcement parallel to the axis of the strut. These bars must be properly anchored and enclosed by ties or spirals per 7.10. The compressive strength of these longitudinally reinforced struts can be calculated as:

$$F_{ns} = f_{ce}A_{cs} + A_s'f_s' \qquad \textit{Eq. (A-5)}$$

where f_s' is the stress in the longitudinal strut reinforcement at nominal strength. It can be either obtained from strain analyses at the time the strut crushes or taken as $f_s' = f_y$ for Grade 40 and 60 rebars.

A.4 STRENGTH OF TIES

The nominal strength of a tie is calculated as the sum of yield strength of the conventional reinforcement plus the force in the prestressing steel:

$$F_{nt} = A_{ts}f_y + A_{tp}\left(f_{se} + \Delta f_p\right) \qquad \textit{Eq. (A-6)}$$

Note, that A_{tp} is zero if there is no prestressing present in the tie. The actual prestressing stress $(f_{se} + \Delta f_p)$ should not exceed the yield stress f_{py} of the prestressing steel. Also, if not calculated, the code allows to estimate the increase in prestressing steel stress due to factored loads Δf_p, as 60,000 psi for bonded prestressed reinforcement, or 10,000 psi for unbonded prestressed reinforcement.

Since the intent of having a tie is to provide for a tension element in a truss, the axis of the reinforcement centroid shall coincide with the axis of the tie assumed in the model. Depending on the distribution of the tie reinforcement, the effective tie width (w_t) may vary between the following limits:

- The minimum width for configurations where only one layer of reinforcement provided in a tie, w_t can be taken as the diameter of the bars in the tie plus twice the concrete cover to the surface of the ties. Should the tie be wider than this, the reinforcement shall be distributed evenly over the width.
- The upper limit is established as the width corresponding to the width in a hydrostatic nodal zone, calculated as

  $$w_{t,max} = F_{nt}/f_{ce}$$

  where f_{ce} is the applicable effective compression strength of a nodal zone discussed below.

Nodes shall be able to develop the difference between the forces of truss members connecting to them. Thus, besides providing adequate amount of tie reinforcement, special attention shall be paid to proper anchorage. Anchorage can be achieved using mechanical devices, post-tensioning anchorage devices, standard hooks, or straight bar embedment. The reinforcement in a tie should be anchored before it leaves the extended nodal zone, i.e., at the point defined by the intersection of the centroid of the bars in the tie and the extensions of the outlines of either the strut or the bearing area as shown on Fig.17-9. For truss layouts where more than one tie intersect at a node, each tie force shall be developed at the point where the centroid of the reinforcement in the tie leaves the extended nodal zone. (Note, that transverse reinforcement required by A3.3 shall be anchored according to the provisions of 12.13).

In many cases the structural configuration does not allow to provide for the straight development length for a tie.

For such cases, anchorage is provided through mechanical devices, hooks, or splicing with several layers of smaller bars. These options often require a wider structural member and/or additional confinement reinforcement (e.g., to avoid cracking along the outside of the hooks).

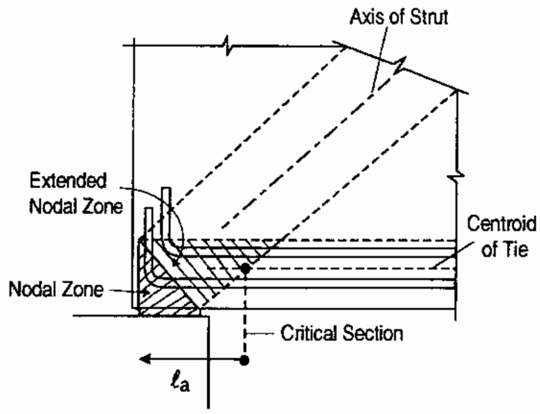

Figure 17-9 Anchorage of Tie Reinforcement

A.5 STRENGTH OF NODAL ZONES

The nominal compression strength at the face of a nodal zone or at any section through the nodal zone shall be:

$$F_{nn} = f_{ce} A_{nz} \qquad \textit{Eq. (A-7)}$$

where A_{nz} is taken as the area of the face of the nodal zone that the strut force F_u acts on, if the face is perpendicular to the line of action of F_u. If the nodal zone is limited by some other criteria, the node-to-strut interface may not be perpendicular to the axis of the strut, therefore, the axial stresses in the compression-only strut will generate both shear and normal stresses acting on the interface. In those cases, the A_{nz} parameter shall be the area of a section, taken through the nodal zone perpendicular to the strut axis.

The strut-and-tie model is applicable to three-dimensional situations as well. In order to keep calculations simple, A5.3 allows the area of the nodal faces to be less than that described above. The shape of each face of the nodal zones must be similar to the shape of the projection of the end of the struts onto the corresponding faces of the nodal zones.

The effective compressive strength of the concrete in the nodal zone (f_{ce}) is calculated as:

$$f_{ce} = 0.85 \beta_n f_c' \qquad \textit{Eq. (A-8)}$$

and must not exceed the effective concrete compressive strength on the face of a nodal zone due to the strut-and-tie model forces, unless confining reinforcement is provided within the nodal zone and its effect is evidenced by tests and analysis. The sign of forces acting on the node influences the capacity at the nodal zones as reflected by the β_n value. The presence of tensile stresses due to ties decreases the nodal zone concrete strength.

β_n = 1.0 in nodal zones bounded by struts or bearing areas (e.g., C-C-C nodes)
β_n = 0.8 in nodal zones anchoring one tie (e.g., C-C-T nodes)
β_n = 0.6 in nodal zones anchoring two or more ties (e.g., C-T-T or T-T-T nodes).

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Example 17.1—Design of Deep Flexural Member by the Strut-and-Tie Model

Determine the required reinforcement for the simply supported transfer girder shown in Fig. 17-10. The single column at midspan subjects the girder to 180 kips dead load and 250 kips live load.

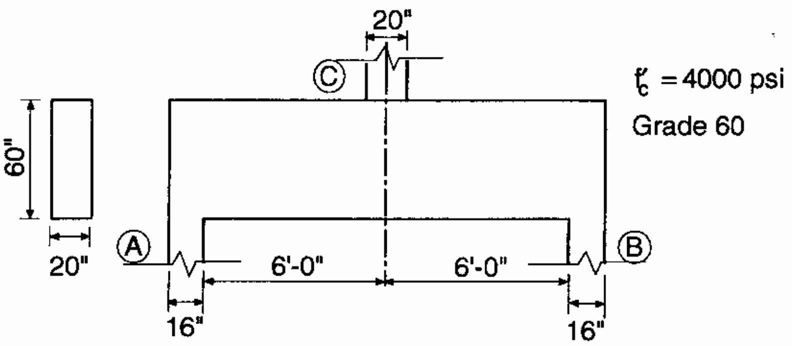


Figure 17-10 Transfer Girder

Calculations and Discussion	Code Reference

1. Calculate factored load and reactions

The transfer girder dead load is conservatively lumped to the column load at midspan. Transfer girder dead load is:

5(20/12) [6 + 6 + (32/12)] 0.15 = 18.5 kips

$P_u = 1.2D + 1.6L = 1.2 \times (18.5 + 180) + 1.6 \times 250 = 640$ kips *Eq. (9-2)*

$R_A = R_B = 640/2 = 320$ kips

2. Determine if this beam satisfies the definition of a "deep beam" *10.7.1* *11.8.1*

Overall girder height h = 5 ft

Clear span $\ell_n = 12$ ft

$$\frac{\ell_n}{h} = \frac{12}{5} = 2.4 < 4$$

Member is a "deep beam" and will be designed using Appendix A.

3. Check the maximum shear capacity of the cross section

$V_u = 320$ kips

Maximum $\phi V_n = \phi\left(10\sqrt{f'_c}b_w d\right)$ *11.8.3*

$= 0.75(10\sqrt{4000} \times 20 \times 54)/1000 = 512 \text{ kips} > V_u$ O.K.

4. Establish truss model

Assume that the nodes coincide with the centerline of the columns (supports), and are located 5 in. from the upper or lower edge of the beam as shown in Fig. 17-11. The strut-and-tie model consists of two struts (A-C and B-C), one tie (A-B), and three nodes (A, B, and C). In addition, columns at A and B act as struts representing reactions. The vertical strut at the top of Node C represents the applied load.

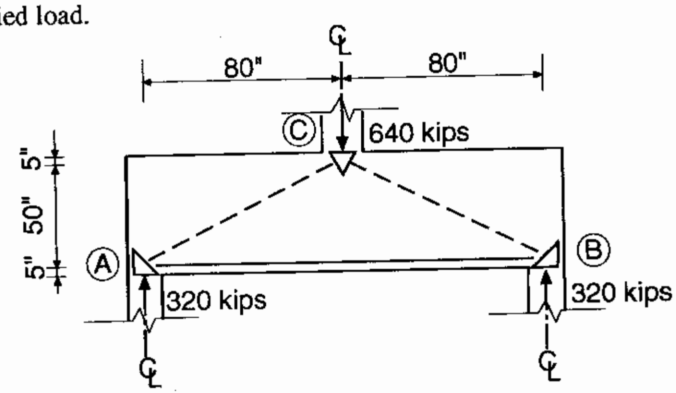


Figure 17-11 Preliminary Truss Layout

The length of the diagonal struts $= \sqrt{50^2 + 80^2} = 94.3$ in.

The force in the diagonal struts $= 320\dfrac{94.3}{50} = 603$ kips

The force in the horizontal tie $= 320\dfrac{80}{50} = 512$ kips

Verify the angle between axis of strut and tie entering Node A.

The angle between the diagonal struts and the horizontal tie $= \tan^{-1}(50/80) = 32° > 25°$ O.K. *A.2.5*

5. Calculate the effective concrete strength (f_{ce}) for the struts assuming that reinforcement is provided per A.3.3 to resist splitting forces. (See Step 9)

For the "bottle-shaped" Struts A-C & B-C

$f_{ce} = 0.85\,\beta_s f'_c = 0.85 \times 0.75 \times 4000 = 2550$ psi *Eq. (A-3)*

where $\beta_s = 0.75$ per A.3.2.2(a)

Note, this effective compressive strength cannot exceed the strength of the nodes at both ends of the strut. See A.3.1.

The vertical struts at A, B, and C have uniform cross-sectional area throughout their length.

$\beta_s = 1.0$ — A.3.2.1

$f_{ce} = 0.85 \times 1.0 \times 4000 = 3400$ psi

6. Calculate the effective concrete strength (f_{ce}) for Nodal Zones A, B, and C

Nodal Zone C is bounded by three struts. So this is a C-C-C nodal zone
with $\beta_n = 1.0$ — A.5.2.1

$f_{ce} = 0.85\beta_n f'_c = 0.85 \times 0.80 \times 4000 = 2720$ psi — Eq. (A-8)

Nodal Zones A and B are bounded by two struts and a tie. For a C-C-T node:

$\beta_n = 0.80$

$f_{ce} = 0.85\beta_n f'_c = 0.85 \times 0.80 \times 4000 = 2720$ psi

7. Check strength at Node C

Assume that a hydrostatic nodal zone is formed at Node C. This means that the faces of the nodal zone are perpendicular to the axis of the respective struts, and that the stresses are identical on all faces.

To satisfy the strength criteria for all three struts and the node, the minimum nodal face dimension is determined based on the least strength value of $f_{ce} = 2550$ psi. The same strength value will be used for Nodes A and B as well.

The strength checks for all components of the strut and tie model are based on
$\phi F_n \geq F_u$ — Eq. (A-1)

where $\phi = 0.75$ for struts, ties, and nodes. — 9.3.2.6

The length of the horizontal face of Nodal Zone C is calculated as

$$\frac{640{,}000}{0.75 \times 2550 \times 20} = 16.7 \text{ in. (less than column width of 20 in.)}$$

The length of the other faces, perpendicular to the diagonal struts, can be obtained from proportionality:

$$16.7 \times \frac{603}{640} = 15.7 \text{ in.}$$

8. Check the truss geometry.

At Node C

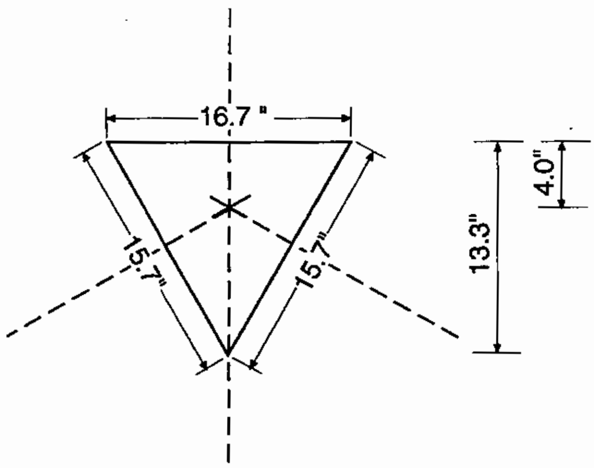


Figure 17-12 Geometry of Node C

The center of the nodal zone is at 4.0 in. from the top of the beam, which is very close to the assumed 5 in.

At Node A

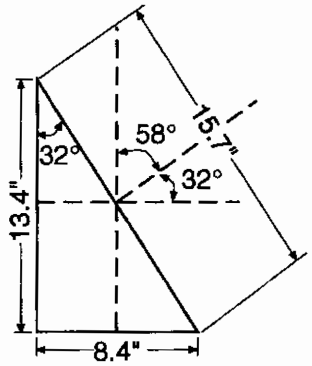


Figure 17-13 Geometry of Node A

The horizontal tie should exert a force on this node to create a stress of 2550 psi. Thus size of the vertical face of the nodal zone is

$$\frac{512{,}000}{0.75 \times 2550 \times 20} = 13.4 \text{ in.}$$

The center of the tie is located 13.4/2 = 6.7 in. from the bottom of the beam. This is reasonably close to the 5 in. originally assumed, so no further iteration is warranted.

Width of node at Support A

$$\frac{320{,}000}{0.75 \times 2550 \times 20} = 8.4 \text{ in.}$$

9. Provide vertical and horizontal reinforcement to resist splitting of diagonal struts.

The angle between the vertical ties and the struts is 90°- 32° = 58° (sin 58° = 0.85).

Try two overlapping No. 4 ties @ 12 in. O.C. (to accomodate the longitudinal tie reinforcement designed in Step 10, below.

$$\frac{A_{si}}{bs}\sin\gamma_i = \frac{4 \times 0.20}{20 \times 12} \times 0.85 = 0.00283$$ *Eq. (A-4)*

and No. 5 horizontal bars @ 12 in. O.C. on each side face (sin 32° = 0.53)

$$\frac{2 \times 0.31}{20 \times 12} \times 0.53 = 0.00137$$

$$\Sigma\frac{A_{si}}{bs}\sin\gamma_i = 0.00283 + 0.00137 = 0.0042 > 0.003 \text{ O.K.}$$ *Eq. (A-4)*

10. Provide horizontal reinforcing steel for the tie

$$A_{s,req} = \frac{F_u}{\phi f_y} = \frac{512}{0.75 \times 60} = 11.4 \text{ in.}^2$$

Select 16 - No. 8 A_s = 12.64 in.²

These bars must be properly anchored. The anchorage is to be measured from the point where the tie exits the extended nodal zone as shown in Fig. 17-14.

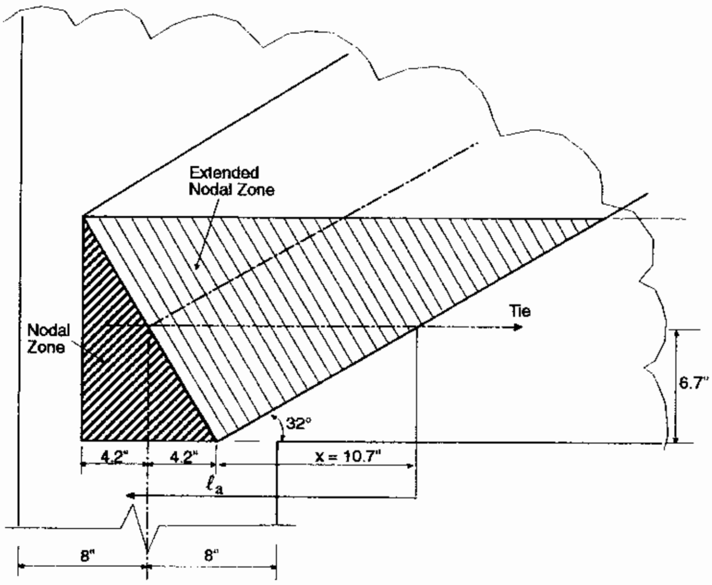


Figure 17-14 Development of Tie Reinforcement Within the Extended Nodal Zone

Distance $x = 6.7/\tan 32 = 10.7$ in.

Available space for a straight bar embedment

10.7 + 4.2 + 8 - 2.0 (cover) = 20.9 in.

This length is inadequate to develop a straight No. 8 bar.

Development length for a No. 8 bar with a standard 90 deg. hook

$$\ell_{dh} = (0.02\psi_e \lambda f_y / \sqrt{f'_c})\, d_b$$ *12.5.2*

$$= \left(0.02(1.0)(1.0)60{,}000/\sqrt{4000}\right) 1.0$$

$$= 19.0 < 20.9 \text{ in. O.K.}$$

Note: the 90 degree hooks will be enclosed within the column reinforcement that extends in the transfer girder. (Fig. 17-15) By providing adequate cover and transverse confinement, the development length of the standard hook could be reduced by the modifiers of 12.5.3.

Less congested reinforcement schemes can be devised with reinforcing steel welded to bearing plates, or with the use of prestressing steel.

Comments:

The discrepancy in the vertical location of the nodes results in a negligible (about 1.5 percent) difference in the truss forces. Thus, another iteration is not warranted.

There are several alternative strut-and-tie models that could have been selected for this problem. An alternative truss layout is illustrated in Fig. 17-16. It has the advantage that the force in the bottom chord varies between nodes, instead of being constant between supports. Further, the truss posts carry truss forces, instead of providing vertical reinforcement just for crack control (A.3.3.1). Finally, the diagonals are steeper, therefore the diagonal compression and the bottom chord forces are reduced. The optimum idealized truss is one that requires the least amount of reinforcement.

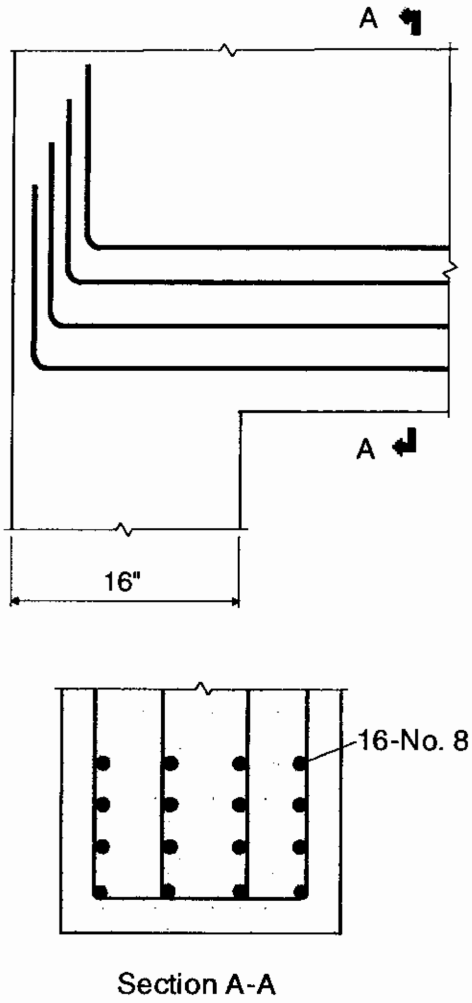


Figure 17-15 Detail of Tie Reinforcement

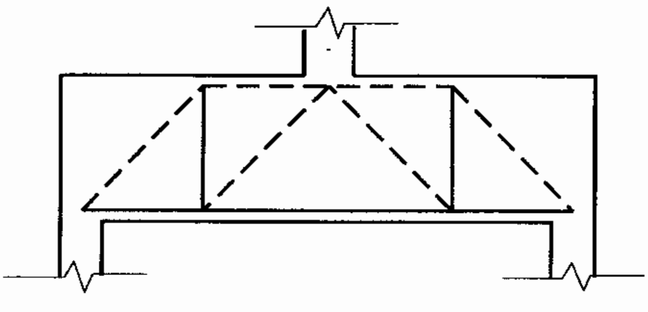


Figure 17-16 Alternative Strut-and-Tie Model

Example 17.2—Design of Column Corbel

Design the single corbel of the 16 in. × 16 in. reinforced concrete column for a vertical force V_u = 60 kips and horizontal force N_u = 12 kips. Assume f_c' = 5000 psi, and Grade 60 reinforcing steel.

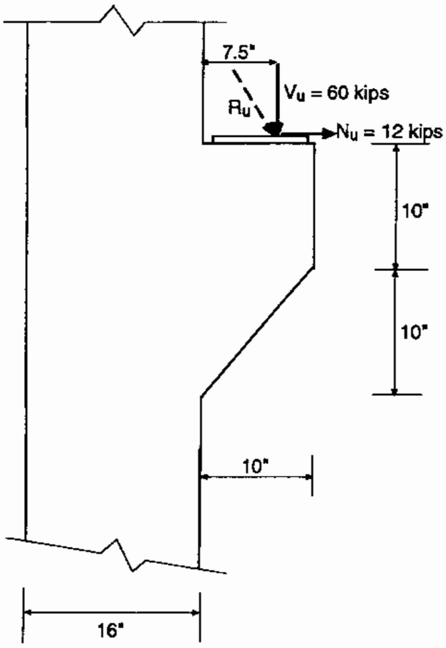


Figure 17-17 Design of Corbel

Calculations and Discussion	Code Reference

1. Establish the geometry of a trial truss and calculate force demand in members.

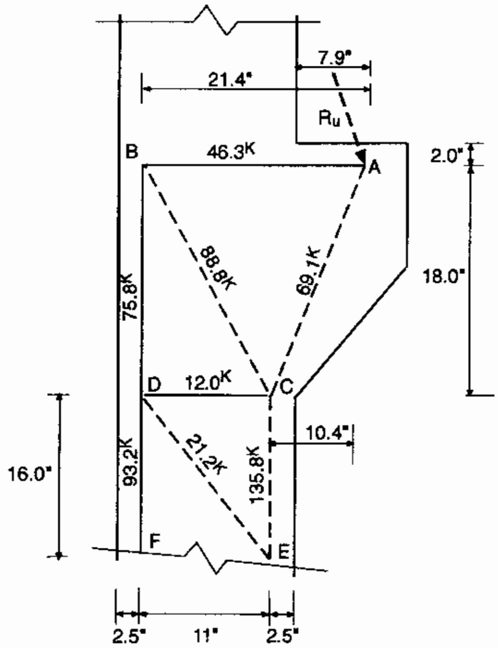


Figure 17-18 Truss Layout

2. Provide reinforcement for ties

Use $\phi = 0.75$ *9.3.2.6*

The nominal strength of ties is to be taken as:

$$F_{nt} = A_{ts}f_y + A_{tp}\left(f_{se} + \Delta f_p\right)$$ *Eq. (A-6)*

where the last term can be ignored for nonprestressed reinforcement

Tie AB $F_u = 46.3$ kips

$$A_{ts} = \frac{F_u}{\phi f_y} = \frac{46.3}{0.75 \times 60} = 1.03 \text{ in.}^2$$ Provide 4 - No. 5 $A_{ts} = 1.24$ in.^{2}

Tie CD $F_u = 12.0$ kips

$$A_{ts} = \frac{12.0}{0.75 \times 60} = 0.27 \text{ in.}^2$$ Provide No. 4 tie (2 legs) $A_{ts} = 0.40$ in.^{2}

Tie BD & DF $P_u = 93.2$ kips

$$A_{ts} = \frac{93.2}{0.75 \times 60} = 2.07 \text{ in.}^2$$ Provide steel in addition of the vertical column reinforcement

This reinforcement may be added longitudinal bar or a rebar bent at Node A, that is used as Tie AB as well.

3. Calculate strut widths

It is assumed that transverse reinforcement will be provided in compliance with A.3.3, so a $\beta_s = 0.75$ can be used in calculating the strut length

$$f_{ce} = 0.85\,\beta_s f_c' = 0.85 \times 0.75 \times 5000 = 3187 \text{ kips}$$ *Eq. (A-3)*

$$\phi f_{ce} = 0.75 \times 3187 = 2390 \text{ psi}$$

Calculate the width of struts required

Strut AC $P_u = 69.1$ kips

$$w = \frac{69{,}100}{16 \times 2390} = 1.81 \text{ in.}$$

Strut BC

$$w = \frac{88{,}800}{16 \times 2390} = 2.32 \text{ in.}$$

Strut CE

$$w = \frac{135{,}800}{16 \times 2390} = 3.55 \text{ in.}$$

Strut DE

$$w = \frac{21.2}{16 \times 2390} = 0.55 \text{ in.}$$

The width of the struts will fit within the concrete column with the corbel.

Provide confinement reinforcement for the struts per A.3.3 in the form of horizontal ties
The angle of the diagonal struts to the horizontal hoops is 58 degree. Provide No. 4 hoops at 4.5 in. on center.

$$\frac{A_s}{bs} \sin\gamma = \frac{2 \times 0.20}{24 \times 4.5} \sin 58° = 0.0031 > 0.003 \quad \text{O.K.}$$

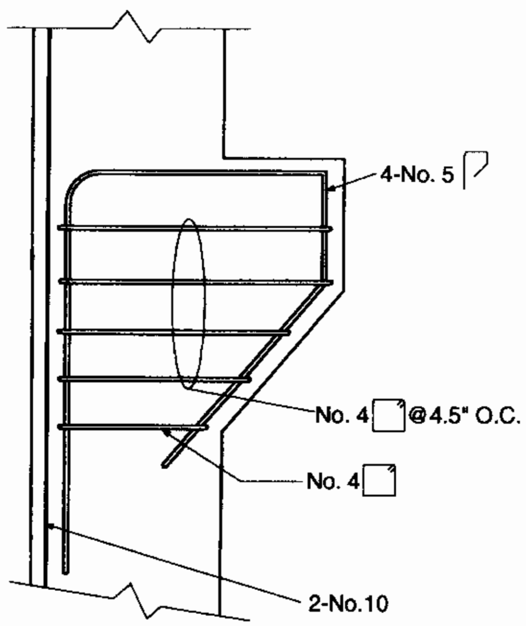


Figure 17-19 Reinforcement Details

Blank

18

Two-Way Slab Systems

UPDATE FOR THE '05 CODE

The primary drop panel definition was moved to Chapter 2. A new Section 13.2.5 was added to give additional dimensional requirements for drop panel if the drop panel is intended to reduce the amount of negative reinforcement over a column or minimum required slab thickness. The new Section 13.2.5 replaces the deleted Sections 13.3.7.1, 13.3.7.2, and 13.3.7.3.

13.1 SCOPE

Figure 18-1 shows the various types of two-way reinforced concrete slab systems in use at the present time that may be designed according to Chapter 13.

A solid slab supported on beams on all four sides (Fig. 18-1(a)) was the original slab system in reinforced concrete. With this system, if the ratio of the long to the short side of a slab panel is two or more, load transfer is predominantly by bending in the short direction and the panel essentially acts as a one-way slab. As the ratio of the sides of a slab panel approaches unity (or as panel approaches a square shape), significant load is transferred by bending in both orthogonal directions, and the panel should be treated as a two-way rather than a one-way slab.

As time progressed and technology evolved, the column-line beams gradually began to disappear. The resulting slab system consisting of solid slabs supported directly on columns is called the flat plate (Fig. 18-1(b)). The two-way flat plate is very efficient and economical and is currently the most widely used slab system for multistory construction, such as motels, hotels, dormitories, apartment buildings, and hospitals. In comparison to other concrete floor/roof systems, flat plates can be constructed in less time and with minimum labor costs because the system utilizes the simplest possible formwork and reinforcing steel layout. The use of flat plate construction also has other significant economic advantages. For instance, because of the shallow thickness of the floor system, story heights are automatically reduced, resulting in smaller overall height of exterior walls and utility shafts; shorter floor-to-ceiling partitions; reductions in plumbing, sprinkler, and duct risers; and a multitude of other items of construction. In cities like Washington, D.C., where the maximum height of buildings is restricted, the thin flat plate permits the construction of the maximum number of stories in a given height. Flat plates also provide for the most flexibility in the layout of columns, partitions, small openings, etc. An additional advantage of flat plate slabs that should not be overlooked is their inherent fire resistance. Slab thickness required for structural purposes will, in most cases, provide the fire resistance required by the general building code, without having to apply spray-on fire proofing, or install a suspended ceiling. This is of particular importance where job conditions allow direct application of the ceiling finish to the flat plate soffit, eliminating the need for suspended ceilings. Additional cost and construction time savings are then possible as compared to other structural systems.

The principal limitation on the use of flat plate construction is imposed by shear around the columns (13.5.4). For heavy loads or long spans, the flat plate is often thickened locally around the columns creating what are known as drop panels. When a flat plate incorporates drop panels, it is called a flat slab (Fig. 18-1(c)). Also for reasons of shear around the columns, the column tops are sometimes flared, creating column capitals. For purposes of design, a column capital is part of the column, whereas a drop panel is part of the slab (13.7.3 and 13.7.4).

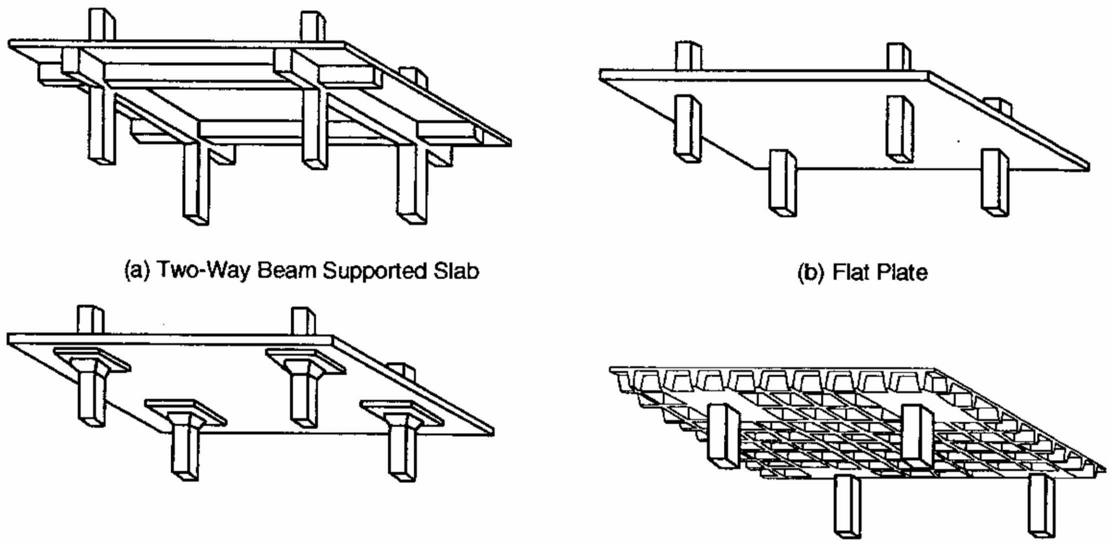


Figure 18-1 Types of Two-Way Slab Systems

Waffle slab construction (Fig. 18-1(d)) consists of rows of concrete joists at right angles to each other with solid heads at the column (needed for shear strength). The joists are commonly formed by using standard square "dome" forms. The domes are omitted around the columns to form the solid heads. For design purposes, waffle slabs are considered as flat slabs with the solid heads acting as drop panels (13.1.3). Waffle slab construction allows a considerable reduction in dead load as compared to conventional flat slab construction since the slab thickness can be minimized due to the short span between the joists. Thus, it is particularly advantageous where the use of long span and/or heavy loads is desired without the use of deepened drop panels or support beams. The geometric shape formed by the joist ribs is often architecturally desirable.

13.1.4 Deflection Control—Minimum Slab Thickness

Minimum thickness/span ratios enable the designer to avoid extremely complex deflection calculations in routine designs. Deflections of two-way slab systems need not be computed if the overall slab thickness meets the minimum requirements specified in 9.5.3. Minimum slab thicknesses for flat plates, flat slabs, and waffle slabs based on Table 9.5(c), and two-way beam-supported slabs based on Eqs. (9-12) and (9-13) are summarized in Table 18-1, where ℓ_n is the clear span length in the long direction of a two-way slab panel. The tabulated values are the controlling minimum thicknesses governed by interior, side, or corner panels assuming a constant slab thickness for all panels making up a slab system. Practical edge beam sizes will usually provide beam-to-slab stiffness ratios α greater than the minimum specified value of 0.8. A "standard" size drop panel that would allow a 10% reduction in the minimum required thickness of a flat slab floor system is illustrated in Fig. 18-2. Note that a drop of larger size and depth may be used if required for shear strength; however, a corresponding lesser slab thickness is not permitted unless deflections are computed.

For design convenience, minimum thicknesses for the six types of two-way slab systems listed in Table 18-1 are plotted in Fig. 18-3.

Refer to Part 10 for a general discussion on control of deflections for two-way slab systems, including design examples of deflection calculations for two-way slabs.

Table 18-1 Minimum Thickness for Two-Way Slab Systems (Grade 60 Reinforcement)

Two-Way Slab System	α_m	β	Minimum h
Flat Plate	—	≤ 2	$\ell_n/30$
Flat Plate with Spandrel Beams[1] [Min. h = 5 in.]	—	≤ 2	$\ell_n/33$
Flat Slab[2]	—	≤ 2	$\ell_n/33$
Flat Slab[2] with Spandrel beams[1] [Min. h = 4 in.]	—	≤ 2	$\ell_n/36$
Two-Way Beam-Supported Slab[3]	≤ 0.2	≤ 2	$\ell_n/30$
	1.0	1	$\ell_n/33$
		2	$\ell_n/36$
	≥ 2.0	1	$\ell_n/37$
		2	$\ell_n/44$
Two-Way Beam-Supported Slab[1,3]	≤ 0.2	≤ 2	$\ell_n/33$
	1.0	1	$\ell_n/36$
		2	$\ell_n/40$
	≥ 2.0	1	$\ell_n/41$
		2	$\ell_n/49$

[1] *Spandrel beam-to-slab stiffness ratio $\alpha \geq 0.8$ (9.5.3.3)*

[2] *Drop panel length $\geq \ell/3$, depth $\geq 1.25h$ (13.3.7)*

[3] *Min. h = 5 in. for $\alpha_m \leq 2.0$; min. h = 3.5 in. for $\alpha_m > 2.0$ (9.5.3.3)*

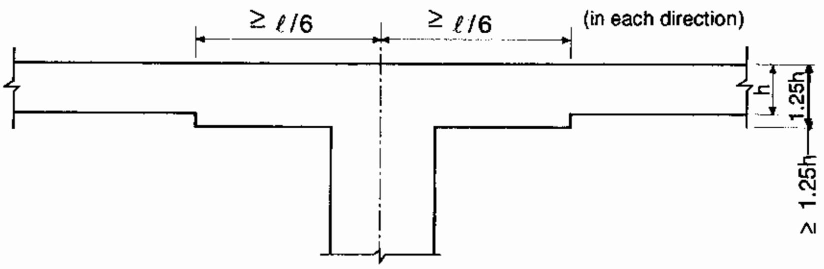


Figure 18-2 Drop Panel Details (13.2.5)

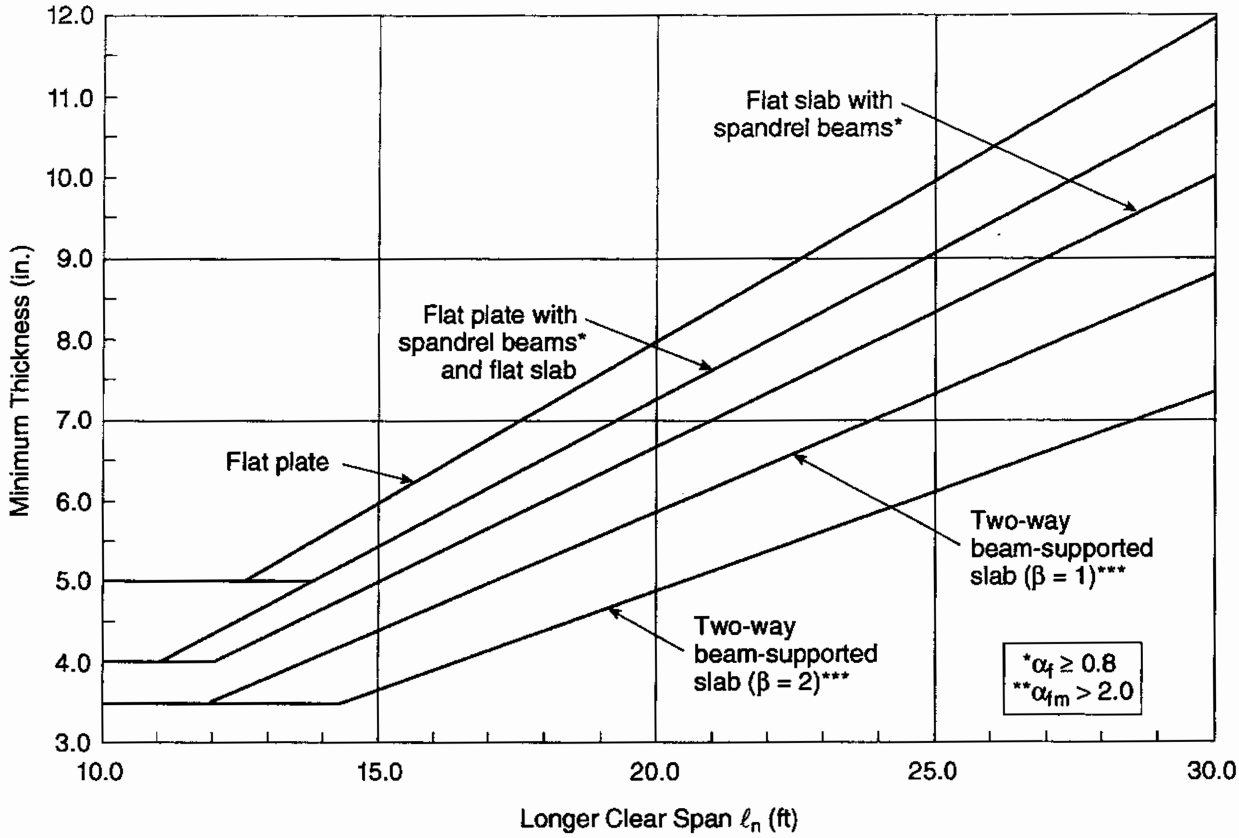


Figure 18-3 Minimum Slab Thickness for Two-Way Slab Systems (see Table 18-1)

13.2 DEFINITIONS

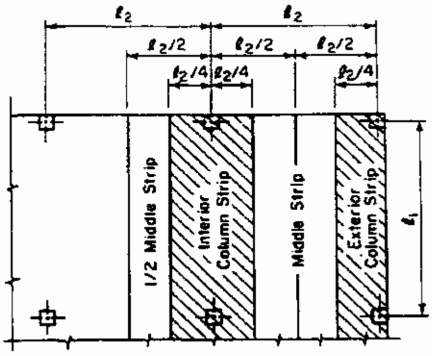
13.2.1 Design Strip

For analysis of a two-way slab system by either the Direct Design Method (13.6) or the Equivalent Frame Method (13.7), the slab system is divided into design strips consisting of a column strip and half middle strip(s) as defined in 13.2.1 and 13.2.2, and as illustrated in Fig. 18-4. The column strip is defined as having a width equal to one-half the transverse or longitudinal span, whichever is smaller. The middle strip is bounded by two column strips. Some judgment is required in applying the definitions given in 13.2.1 for column strips with varying span lengths along the design strip.

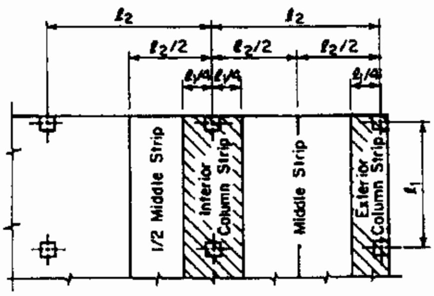
The reason for specifying that the column strip width be based on the shorter of ℓ_1 or ℓ_2 is to account for the tendency for moment to concentrate about the column line when the span length of the design strip is less than its width.

13.2.4 Effective Beam Section

For slab systems with beams between supports, the beams include portions of the slab as flanges, as shown in Fig. 18-5. Design constants and stiffness parameters used with the Direct Design and Equivalent Frame analysis methods are based on the effective beam sections shown.



(a) Column Strip for $\ell_2 \leq \ell_1$



(b) Column Strip for $\ell_2 > \ell_1$

Figure 18-4 Definition of Design Strips

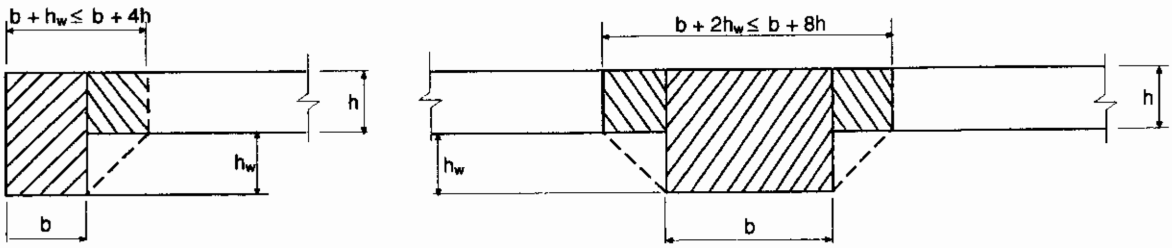


Figure 18-5 Effective Beam Section (13.2.4)

13.3 SLAB REINFORCEMENT

- Minimum area of reinforcement in each direction for two-way slab systems = 0.0018bh (b = slab width, h = total thickness) for Grade 60 bars for either top or bottom steel (13.3.1).
- Maximum bar spacing is 2h, but not more than 18 in. (13.3.2).
- Minimum extensions for reinforcement in slabs without beams (flat plates and flat slabs) are prescribed in Fig. 13.3.8 (13.3.8.1).

Note that the reinforcement details of Fig. 13.3.8 do not apply to two-way slabs with beams between supports or to slabs in non-sway or sway frames resisting lateral loads. For those slabs, a general analysis must be made according to Chapter 12 of the Code to determine bar lengths based on the moment variation but shall not be less than those prescribed in Fig. 13.3.8 (13.3.8.4). Reinforcement details for bent bars were deleted from Fig. 13.3.8 in the '89 code in view of their rare usage in today's construction. Designers who wish to use bent bars in two-way slabs (without beams) should refer to Fig. 13.4.8 of the '83 code, with due consideration of the integrity requirements of 7.13 and 13.3.8 in the current code.

According to 13.3.6, special top and bottom reinforcement must be provided at the exterior corners of a slab with spandrel beams that have a value of α greater than 1.0. The reinforcement must be designed for a moment equal to the largest positive moment per unit width in the panel, and must be placed in a band parallel to the diagonal in the top of the slab and a band perpendicular to the diagonal in the bottom of the slab (Fig. 18-6 (a)); alternatively, it may be placed in two layers parallel to the edges of the slab in both the top and bottom of the slab (Fig. 18-6 (b)). Additionally, the reinforcement must extend at least one-fifth of the longer span in each direction from the corner.

In slabs without beams, all bottom bars in the column strip shall be continuous or spliced with class A splices or with mechanical or welded splices satisying 12.14.3 (13.3.8.5) to provide some capacity for the slab to span to an adjacent support in the event a single support is damaged. Additionally, at least two of these continuous bottom bars shall pass through the column and be anchored at exterior supports. In lift-slab construction and slabs with shearhead reinforcement, clearance may be inadequate and it may not be practical to pass the column strip bottom reinforcing bars through the column. In these cases, two continuous bonded bottom bars in each direction shall pass as close to the column as possible through holes in the shearhead arms or, in the case of lift-slab construction, within the lifting collar (13.3.8.6). This condition was initially addressed in the 1992 Code and was further clarified in 1999.

13.4 OPENINGS IN SLAB SYSTEMS

The code permits openings of any size in any slab system, provided that an analysis is performed that demonstrates that both strength and serviceability requirements are satisfied (13.4.1). For slabs without beams; the analysis of 13.4.1 is waived when the provisions of 13.4.2.1 through 13.4.2.4 are met:

- In the area common to intersecting middle strips, openings of any size are permitted (13.4.2.1).
- In the area common to intersecting column strips, maximum permitted opening size is one-eighth the width of the column strip in either span (13.4.2.2).
- In the area common to one column strip and one middle strip, maximum permitted opening size is limited such that only a maximum of one-quarter of slab reinforcement in either strip may be interrupted (13.4.2.3).

The total amount of reinforcement required for the panel without openings, in both directions, shall be maintained; thus, reinforcement interrupted by the opening must be replaced on each side of the opening. Figure 18-7 illustrates the provisions of 13.4.2 for slabs with $\ell_2 > \ell_1$. Refer to Part 16 for a discussion on the effect of openings in slabs without beams on concrete shear strength (13.4.2.4).

13.5 DESIGN PROCEDURES

Section 13.5.1 permits design (analysis) of two-way slab systems by any method that satisfies code-defined strength requirements (9.2 and 9.3), and all applicable code serviceability requirements, including specified limits on deflections (9.5.3).

13.5.1.1 Gravity Load Analysis—Two methods of analysis of two-way slab systems under gravity loads are addressed in Chapter 13: the simpler Direct Design Method (DDM) of 13.6, and the more complex Equivalent Frame Method (EFM) of 13.7. The Direct Design Method is an approximate method using moment coeffi cients, while the Equivalent Frame (elastic analysis) Method is more exact. The approximate analysis procedure

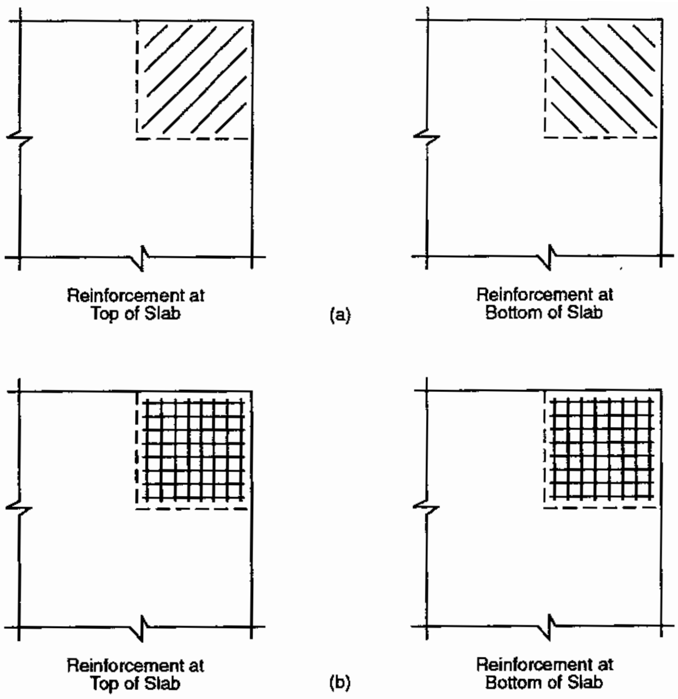


Figure 18-6 Special Reinforcement Required at Corners of Beam-Supported Slabs

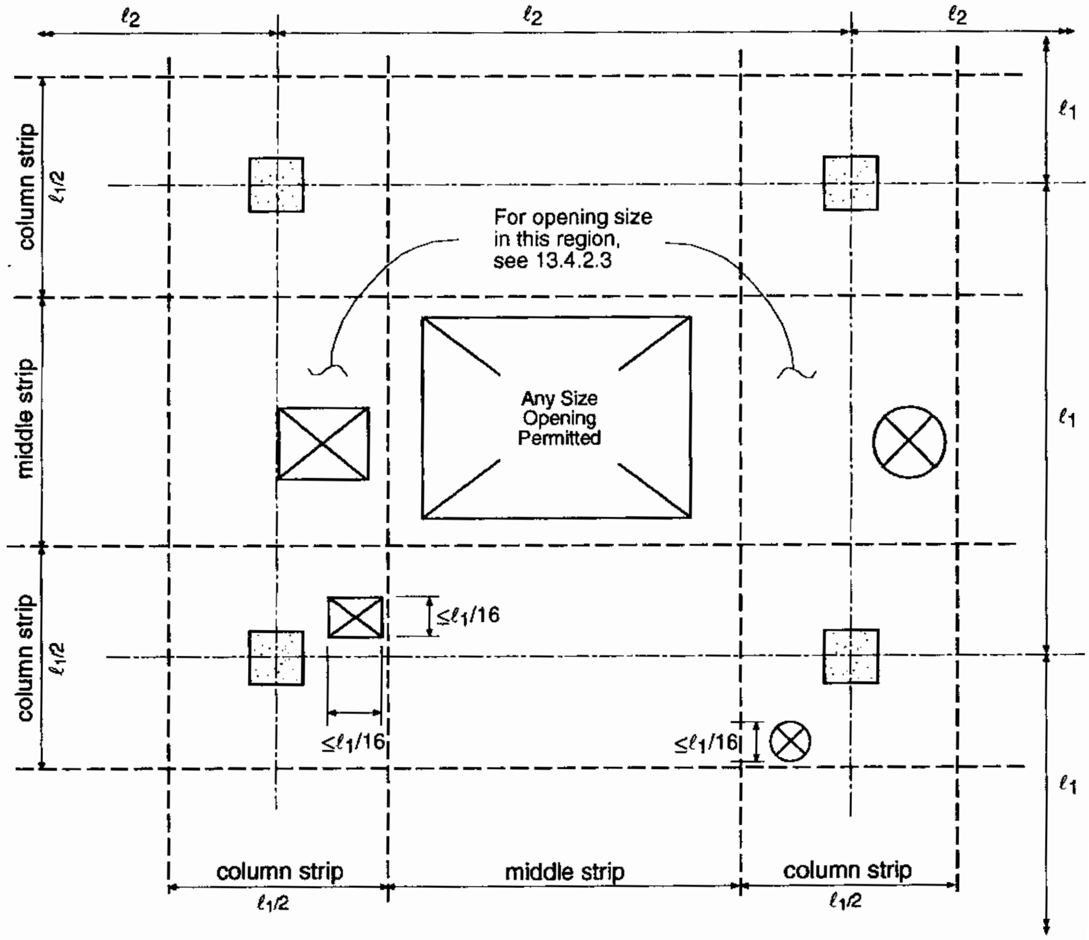


Figure 18-7 Permitted Openings in Slab Systems without Beams for $\ell_2 > \ell_1$

of the Direct Design Method will give reasonably conservative moment values for the stated design conditions for slab systems within the limitations of 13.6.1.

Both methods are for analysis under gravity loads only, and are limited in application to buildings with columns and/or walls laid out on a basically orthogonal grid, i.e., where column lines taken longitudinally and transversely through the building are mutually perpendicular. Both methods are applicable to slabs with or without beams between supports. Note that neither method applies to slab systems with beams spanning between other beams; the beams must be located along column lines and be supported by columns or other essentially nondeflecting supports at the corners of the slab panels.

13.5.1.2 Lateral Load Analysis—For lateral load analysis of frames, the model of the structure may be based upon any approach that is shown to satisfy equilibrium and geometric compatibility and to be in reasonable agreement with test data. Acceptable approaches include plate-bending finite-element models, effective beam width models, and equivalent frame models. The stiffness values for frame members used in the analysis must reflect effects of slab cracking, geometric parameters, and concentration of reinforcement.

During the life of the structure, ordinary occupancy loads and volume changes due to shrinkage and temperature effects will cause cracking of slabs. To ensure that lateral drift caused by wind or earthquakes is not underestimated, cracking of slabs must be considered in stiffness assumptions for lateral drift calculations.

The stiffness of slab members is affected not only by cracking, but also by other parameters such as ℓ_2/ℓ_1, c_1/ℓ_1, c_2/c_1, and on concentration of reinforcement in the slab width defined in 13.5.3.2 for unbalanced moment transfer by flexure. This added concentration of reinforcement increases stiffness by preventing premature yielding and softening in the slab near the column supports. Consideration of the actual stiffness due to these factors is important for lateral load analysis because lateral displacement can significantly affect the moments in the columns, especially in tall moment frame buildings. Also, actual lateral displacement for a single story, or for the total height of a building is an important consideration for building stability and performance.

Cracking reduces stiffness of the slab-beams as compared with that of an uncracked floor. The magnitude of the loss of stiffness due to cracking will depend on the type of slab system and reinforcement details. For example, prestressed slab systems with reduced slab cracking due to prestressing, and slab systems with large beams between columns will lose less stiffness than a conventional reinforced flat plate system.

Prior to the 1999 code, the commentary indicated stiffness values based on Eq. (9-8) were reasonable. However, this was deleted from the commentary in 1999, since factors such as volume change effects and early age loading are not adequately represented in Eq. (9-8) . Since it is difficult to evaluate the effect of cracking on stiffness, it is usually sufficient to use a lower bound value. On the assumption of a fully cracked slab with minimum reinforcement at all locations, a stiffness for the slab-beam equal to one-fourth that based on the gross area of concrete ($K_{sb}/4$) should be reasonable. A detailed evaluation of the effect of cracking may also be made. Since slabs normally have more than minimum reinforcement and are not fully cracked, except under very unusual conditions, the one-fourth value should be expected to provide a safe lower bound for stiffness under lateral loads. See R13.5.1.2 for guidance on stiffness assumption for lateral load analysis.

Moments from an Equivalent Frame (or Direct Design) analysis for gravity loading may be combined with moments from a lateral load analysis (13.5.1.3). Alternatively, the Equivalent Frame Analysis can be used for lateral load analysis, if modified to account for reduced stiffness of the slab-beams.

For both vertical and lateral load analyses, moments at critical sections of the slab-beams are transversely distributed in accordance with 13.6.4 (column strips) and 13.6.6 (middle strips).

13.5.4 Shear in Two-Way Slab Systems

If two-way slab systems are supported by beams or walls, the slab shear is seldom a critical factor in design, as the shear force at factored loads is generally well below the shear strength of the concrete.

In contrast, when two-way slabs are supported directly by columns as in flat plates or flat slabs, shear around the columns is of critical importance. Shear strength at an exterior slab-column connection (without edge beams) is especially critical because the total exterior negative slab moment must be transferred directly to the column. This aspect of two-way slab design should not be taken lightly by the designer. Two-way slab systems will normally be found to be quite "forgiving" if an error in the distribution or even in the amount of flexural reinforcement is made, but there will be no forgiveness if the required shear strength is not provided.

For slab systems supported directly by columns, it is advisable at an early stage in design to check the shear strength of the slab in the vicinity of columns as illustrated in Fig. 18-8.

Two types of shear need to be considered in the design of flat plates or flat slabs supported directly on columns. The first is the familiar one-way or beam-type shear, which may be critical in long narrow slabs. Analysis for beam shear considers the slab to act as a wide beam spanning between the columns. The critical section is taken at a distance d from the face of the column. Design against beam shear consists of checking for satisfaction of the requirement indicated in Fig. 18-9(a). Beam shear in slabs is seldom a critical factor in design, as the shear force is usually well below the shear strength of the concrete.

Two-way or "punching" shear is generally the more critical of the two types of shear in slab systems supported directly on columns. Punching shear considers failure along the surface of a truncated cone or pyramid around a column. The critical section is taken perpendicular to the slab at a distance d/2 from the perimeter of a column. The shear force V_u to be resisted can be easily calculated as the total factored load on the area bounded by panel centerlines around the column, less the load applied within the area defined by the critical shear perimeter (see Fig. 18-8).

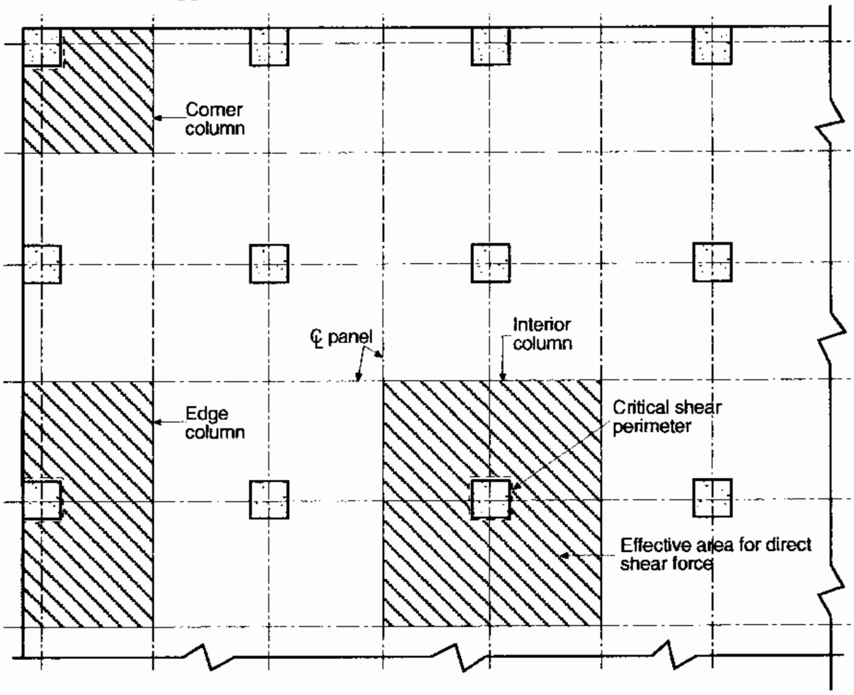


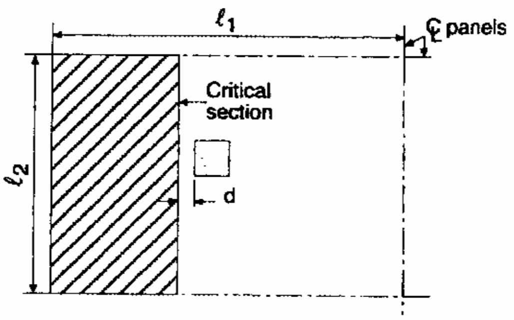
Figure 18-8 Critical Locations for Slab Shear Strength

In the absence of significant moment transfer from the slab to the column, design against punching shear consists of making sure that the requirement of Fig. 18-9(b) is satisfied. For practical design, only direct shear (uniformly distributed around the perimeter b_o) occurs around interior slab-column supports where no (or insignificant) moment is to be transferred from the slab to the column. Significant moments may have to be carried when unbalanced gravity loads on either side of an interior column or horizontal loading due to wind must be transferred from the slab to the column. At exterior slab-column supports, the total exterior slab moment from gravity loads (plus any lateral load moments due to wind or earthquake) must be transferred directly to the column.

13.5.3 Transfer of Moment in Slab-Column Connections

Transfer of moment between a slab and a column takes place by a combination of flexure (13.5.3) and eccentricity of shear (11.12.6.1). Shear due to moment transfer is assumed to act on a critical section at a distance d/2 from the face of the column (the same critical section around the column as that used for direct shear transfer; see Fig. 18-9(b). The portion of the moment transferred by flexure is assumed to be transferred over a width of slab equal to the transverse column width c_2, plus 1.5 times the slab thickness (1.5h) on either side of the column (13.5.3.2). Concentration of negative reinforcement is to be used to resist moment on this effective slab width. The combined shear stress due to direct shear and moment transfer often governs the design, especially at the exterior slab-column supports.

The portions of the total unbalanced moment M_u to be transferred by eccentricity of shear and by flexure are given by Eqs. (11-39) and (13-1), respectively, where $\gamma_v M_u$ is considered transferred by eccentricity of shear, and $\gamma_f M_u$ is considered transferred by flexure. At an interior square column with $b_1 = b_2$, 40% of the moment is transferred by eccentricity of shear ($\gamma_v M_u = 0.40M_u$), and 60% by flexure ($\gamma_f M_u = 0.60M_u$), where M_u is the transfer moment at the centroid of the critical section. The moment M_u at the exterior slab-column support will generally not be computed at the centroid of the critical transfer section. In the Equivalent Frame analysis, moments are computed at the column centerline. In the Direct Design Method, moments are computed at the face of support. Considering the approximate nature of the procedure used to evaluate the stress distribution due to moment transfer, it seems unwarranted to consider a change in moment to the critical section centroid; use of the moment values at column centerline (EFM) or at face of support (DDM) directly would usually be accurate enough.

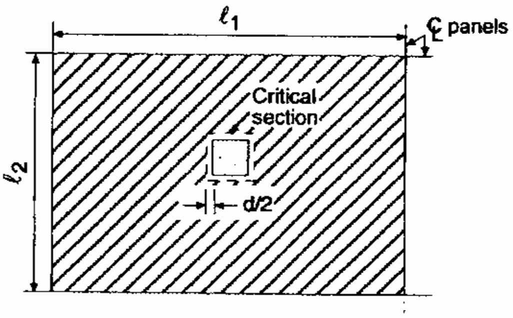


(a) Beam shear

$V_u \le \phi V_c$

$\le \phi 2\sqrt{f'_c}\,\ell_2 d$

where V_u is factored shear force (total factored load on shaded area).



(b) Two-way shear

$V_u \le \phi V_c$

where:

$$\phi V_c = \text{least of} \begin{cases} \phi\left(2 + \dfrac{4}{\beta}\right)\sqrt{f'_c}\,b_o d \\ \phi\left(\dfrac{\alpha_s d}{b_o} + 2\right)\sqrt{f'_c}\,b_o d \\ \phi 4\sqrt{f'_c}\,b_o d \end{cases}$$

V_u = factored shear force (total factored load on shaded area)
b_o = perimeter of critical section
β = long side/short side of reaction area
α_s = constant (11.12.2.1 (b))

Figure 18-9 Direct Shear at an Interior Slab-Column Support (see Fig. 18-8)

The factored shear stress on the critical transfer section is the sum of the direct shear and the shear caused by moment transfer,

$$v_u = \frac{V_u}{A_c} + \frac{\gamma_v M_u c}{J}$$

For slabs supported on square columns, shear stress v_u must not exceed $\phi 4\sqrt{f_c'}$.

Computation of the combined shear stress involves the following properties of the critical transfer section:

A_c = area of critical section

c = distance from centroid of critical section to face of section where stress is being computed

J = property of critical section analogous to polar moment of inertia

The above properties are given in Part 16. Note that in the case of flat slabs, two different critical sections need to be considered in punching shear calculations, as shown in Fig. 18-10.

Unbalanced moment transfer between slab and an edge column (without spandrel beams) requires special consideration when slabs are analyzed by the Direct Design Method for gravity loads. See discussion on 13.6.3.6 in Part 19.

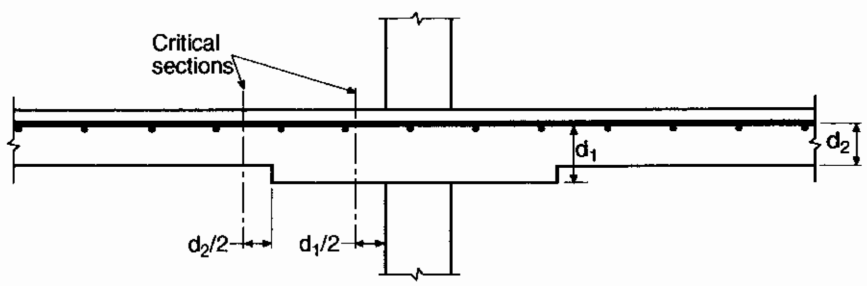


Figure 18-10 Critical Shear-Transfer Sections for Flat Slabs

The provisions of 13.5.3.3 were introduced in the '95 Code. At exterior supports, for unbalanced moments about an axis parallel to the edge, the portion of moment transferred by eccentricity of shear, $\gamma_v M_u$, may be reduced to zero provided that the factored shear at the support (excluding the shear produced by moment transfer) does not exceed 75 percent of the shear strength ϕV_c defined in 11.12.2.1 for edge columns or 50 percent for corner columns. Tests indicate that there is no significant interaction between shear and unbalanced moment at the exterior support in such cases. It should be noted that as $\gamma_v M_u$ is decreased, $\gamma_f M_u$ is increased.

Tests of interior supports have indicated that some flexibility in distributing unbalanced moment by shear and flexure is also possible, but with more severe limitations than for exterior supports. For interior supports, the unbalanced moment transferred by flexure is permitted to be increased up to 25 percent provided that the factored shear (excluding the shear caused by moment transfer) at an interior support does not exceed 40 percent of the shear strength ϕV_c defined in 11.12.2.1.

Note that the above modifications are permitted only when the reinforcement ratio ρ within the effective slab width defined in 13.5.3.2 is less than or equal to $0.375\rho_b$. This provision is intended to improve ductile behavior of the column-slab joint.

SEQUEL

The Direct Design Method and the Equivalent Frame Method for gravity load analysis of two-way slab systems are treated in detail in the following Parts 19 and 20, respectively.

Blank

19

Two-Way Slabs — Direct Design Method

GENERAL CONSIDERATIONS

The Direct Design Method is an approximate procedure for analyzing two-way slab systems subjected to gravity loads only. Since it is approximate, the method is limited to slab systems meeting the limitations specified in 13.6.1. Two-way slab systems not meeting these limitations must be analyzed by more accurate procedures such as the Equivalent Frame Method, as specified in 13.7. See Part 20 for discussion and design examples using the Equivalent Frame Method.

With the publication of ACI 318-83, the Direct Design Method for moment analysis of two-way slab systems was greatly simplified by eliminating all stiffness calculations for determining design moments in an end span. A table of moment coefficients for distribution of the total span moment in an end span (13.6.3.3) replaced the expressions for distribution as a function of the stiffness ratio α_{ec}. As a companion change, the then approximate Eq. (13-4) for unbalanced moment transfer between the slab and an interior column was also simplified through elimination of the α_{ec} term. With these changes, the Direct Design Method became a truly direct design procedure, with all design moments determined directly from moment coefficients. Also, a new 13.6.3.6 was added, addressing a special provision for shear due to moment transfer between a slab without beams and an edge column when the approximate moment coefficients of 13.6.3.3 are used. See discussion on 13.6.3.6 below. Through the 1989 (Revised 1992) edition of the code and commentary, R13.6.3.3 included a "Modified Stiffness Method" reflecting the original distribution, and confirming that design aids and computer programs based on the original distribution as a function of the stiffness ratio α_{ec} were still applicable for usage. The "Modified Stiffness Method" was dropped from R13.6.3.3 in the 1995 edition of the code and commentary.

PRELIMINARY DESIGN

Before proceeding with the Direct Design Method, a preliminary slab thickness h needs to be determined for control of deflections according to the minimum thickness requirements of 9.5.3. Table 18-1 and Fig. 18-3 can be used to simplify minimum thickness computations.

For slab systems without beams, it is advisable at this stage in the design process to check the shear strength of the slab in the vicinity of columns or other support locations in accordance with the special shear provision for slabs (11.12). See discussion on 13.5.4 in Part 18.

Once a slab thickness has been selected, the Direct Design Method, which is essentially a three-step analysis procedure, involves: (1) determining the total factored static moment for each span, (2) dividing the total factored static moment between negative and positive moments within each span, and (3) distributing the negative and the positive moment to the column and the middle strips in the transverse direction.

For analysis, the slab system is divided into design strips consisting of a column strip and two half-middle strip(s) as defined in 13.2.1 and 13.2.2, and as illustrated in Fig. 19-1. Some judgment is required in applying the definitions given in 13.2.1 for slab systems with varying span lengths along the design strip.

13.6.1 Limitations

The Direct Design Method applies within the limitations illustrated in Fig. 19-2:

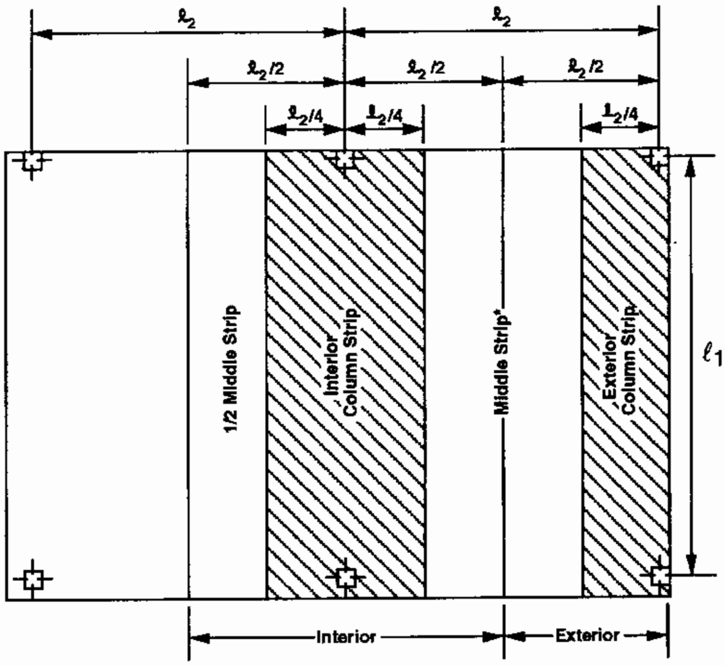
1. There must be three or more continuous spans in each direction;
2. Slab panels must be rectangular with a ratio of longer to shorter span (centerline-to-centerline of supports) not greater than 2;
3. Successive span lengths (centerline-to-centerline of supports) in each direction must not differ by more than 1/3 of the longer span;
4. Columns must not be offset more than 10% of the span (in direction of offset) from either axis between centerlines of successive columns;
5. Loads must be uniformly distributed, with the unfactored or service live load not more than 2 times the unfactored or service dead load ($L/D \leq 2$);
6. For two-way beam-supported slabs, relative stiffness of beams in two perpendicular directions must satisfy the minimum and maximum requirements given in 13.6.1.6; and
7. Redistribution of negative moments by 8.4 is not permitted.

13.6.2 Total Factored Static Moment for a Span

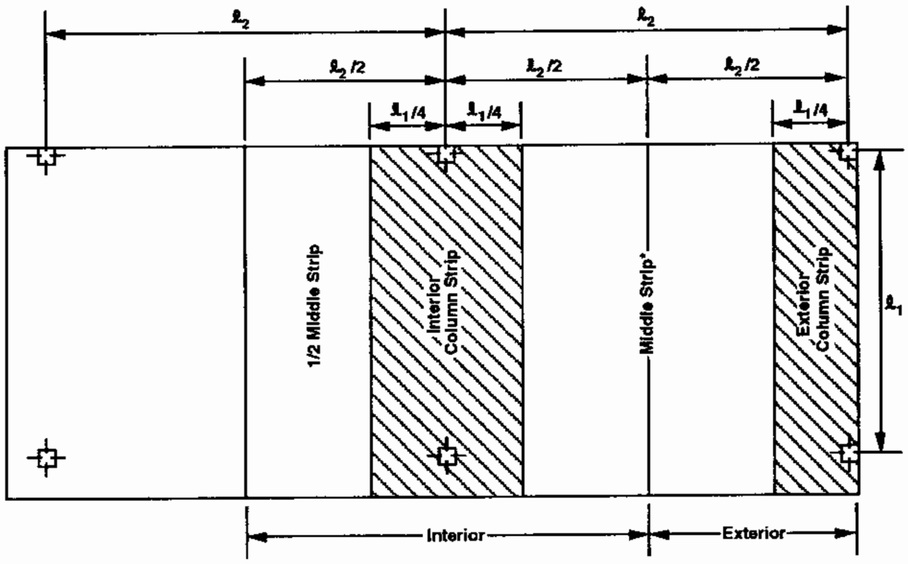
For uniform loading, the total design moment M_o for a span of the design strip is calculated by the simple static moment expression:

$$M_o = \frac{q_u \ell_2 \ell_n^2}{8} \qquad \textit{Eq. (13-4)}$$

where q_u is the factored combination of dead and live loads (psf), $q_u = 1.2w_d + 1.6w_\ell$. The clear span ℓ_n (in the direction of analysis) is defined in a straightforward manner for columns or other supporting elements of rectangular cross-section. The clear span starts at the face of support. Face of support is defined as shown in Fig. 19-3. One limitation requires that the clear span not be taken as less than 65% of the span center-to-center of supports (13.6.2.5). The length ℓ_2 is simply the span (centerline-to-centerline) transverse to ℓ_n; however, when the span adjacent and parallel to an edge is being considered, the distance from edge of slab to panel centerline is used for ℓ_2 in calculation of M_o (13.6.2.4).



(a) Column strip for $\ell_2 \leq \ell_1$



(b) Column strip for $\ell_2 > \ell_1$

* When edge of exterior design strip is supported by a wall, the factored moment resisted by this middle strip is defined in 13.6.6.3.

Figure 19-1 Definition of Design Strips

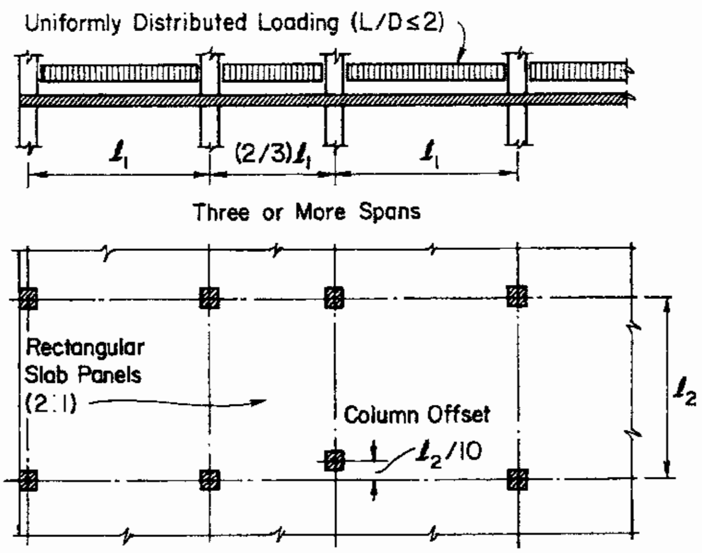
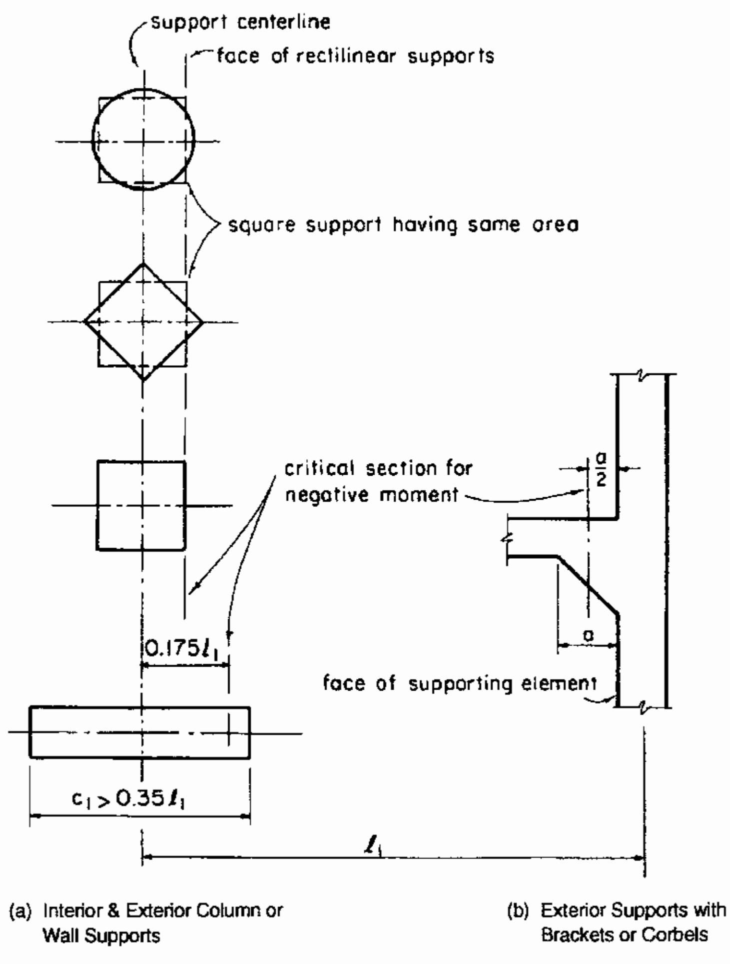


Figure 19-2 Conditions for Analysis by Coefficients



(a) Interior & Exterior Column or Wall Supports

(b) Exterior Supports with Brackets or Corbels

Figure 19-3 Critical Sections for Negative Design Moment

13.6.3 Negative and Positive Factored Moments

The total static moment for a span is divided into negative and positive design moments as shown in Fig. 19-4. End span moments in Fig. 19-4 are shown for a flat plate or flat slab without spandrels (slab system without beams between interior supports and without edge beam). For other end span conditions, the total static moment M_0 is distributed as shown in Table 19-1.

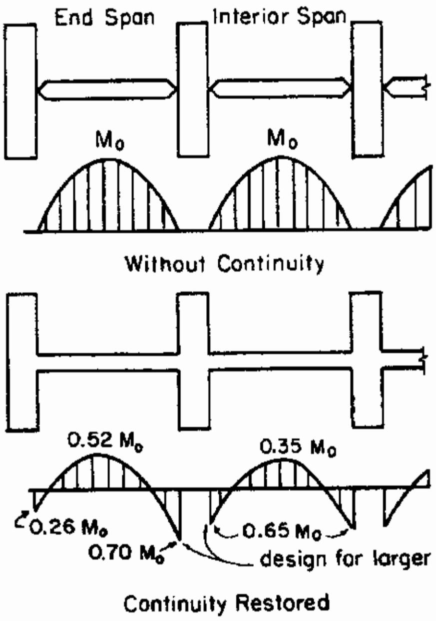


Figure 19-4 Design Strip Moments

Table 19-1 Distribution of Total Static Moment for an End Span

Factored Moment	(1) Slab Simply Supported on Concrete or Masonry Wall	(2) Two-Way Beam-Supported Slabs	(3) Flat Plates and Flat Slabs: Without Edge Beam	(4) Flat Plates and Flat Slabs: With Edge Beam	(5) Slab Monolithic with Concrete Wall
Interior Negative	0.75	0.70	0.70	0.70	0.65
Positive	0.63	0.57	0.52	0.50	0.35
Exterior Negative	0	0.16	0.26	0.30	0.65

13.6.3.6 Special Provision for Load Transfer Between Slab and an Edge Column—For columns supporting a slab without beams, load transfer directly between the slab and the supporting columns (without intermediate load transfer through beams) is one of the more critical design conditions for the flat plate or flat slab system. Shear strength of the slab-column connection is critical. This aspect of two-way slab design should not be taken lightly by the designer. Two-way slab systems are fairly "forgiving" of an error in the distribution or even in the amount of flexural reinforcement; however, there is little or no forgiveness if a critical error in the provision of shear strength is made. See Part 16 for special provisions for direct shear and moment transfer at slab-column connections.

Section 13.6.3.6 addresses the potentially critical moment transfer between a beamless slab and an edge column. To ensure adequate shear strength when using the approximate end-span moment coefficients of 13.6.3.3, the 1989 edition of the code required that the full nominal strength M_n provided by the column strip be used in determining the fraction of unbalanced moment transferred by the eccentricity of shear (γ_v) in accordance with 11.12.6 (for end spans without edge beams, the column strip is proportioned to resist the total exterior negative factored moment). This requirement was changed in ACI 318-95. The moment $0.3M_o$ instead of M_n of the column strip must be used in determining the fraction of unbalanced moment transferred by the eccentricity of shear. The total reinforcement provided in the column strip includes the additional reinforcement concentrated over the column to resist the fraction of unbalanced moment transferred by flexure, $\gamma_f M_u = \gamma_f\,(0.26M_o)$, where the moment coefficient (0.26) is from 13.6.3.3, and γ_f is given by Eq. (13-1).

13.6.4 Factored Moments in Column Strips

The amounts of negative and positive factored moments to be resisted by a column strip, as defined in Fig. 19-1, depends on the relative beam-to-slab stiffness ratio and the panel width-to-length ratio in the direction of analysis. An exception to this is when a support has a large transverse width.

The column strip at the exterior of an end span is required to resist the total factored negative moment in the design strip unless edge beams are provided.

When the transverse width of a support is equal to or greater than three quarters (3/4) of the design strip width, 13.6.4.3 requires that the negative factored moment be uniformly distributed across the design strip.

The percentage of total negative and positive factored moments to be resisted by a column strip may be determined from the tables in 13.6.4.1 (interior negative), 13.6.4.2 (exterior negative) and 13.6.4.4 (positive), or from the following expressions:

Percentage of negative factored moment at interior support to be resisted by column strip

$$= 75 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right) \qquad (1)$$

Percentage of negative factored moment at exterior support to be resisted by column strip

$$= 100 - 10\beta_t + 12\beta_t\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right) \qquad (2)$$

Percentage of positive factored moment to be resisted by column strip

$$= 60 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1.5 - \frac{\ell_2}{\ell_1}\right) \qquad (3)$$

Note: When $\alpha_{f1}\ell_2/\ell_1 > 1.0$, use 1.0 in above equations. When $\beta_t > 2.5$, use 2.5 in Eq. (2) above.

For slabs without beams between supports ($\alpha_{f1} = 0$) and without edge beams ($\beta_t = 0$), the distribution of total negative moments to column strips is simply 75 and 100 percent for interior and exterior supports, respectively, and the distribution of total positive moment is 60 percent. For slabs with beams between supports, distribution depends on the beam-to-slab stiffness ratio; when edge beams are present, the ratio of torsional stiffness of edge beam to flexural stiffness of slab also influences distribution. Figs. 19-6, 19-7, and 19-8 simplify evaluation of the beam-to-slab stiffness ratio α_{f1}. To evaluate β_t, stiffness ratio for edge beams, Table 19-2 simplifies calculation of the torsional constant C.

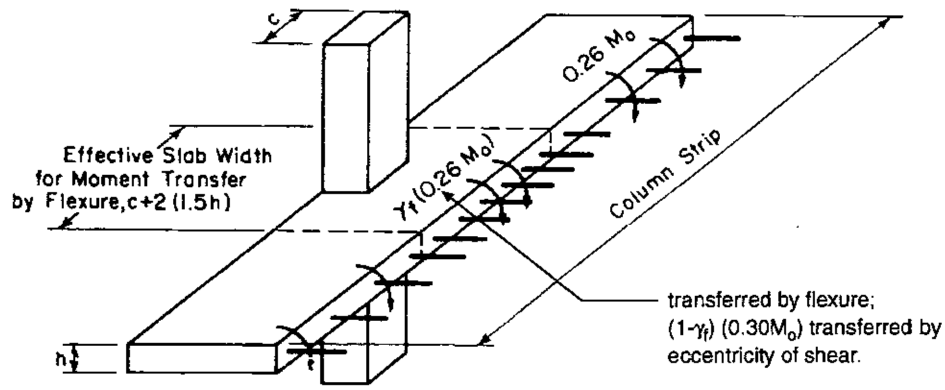


Figure 19-5 Transfer of Negative Moment at Exterior Support Section of Slab without Beams

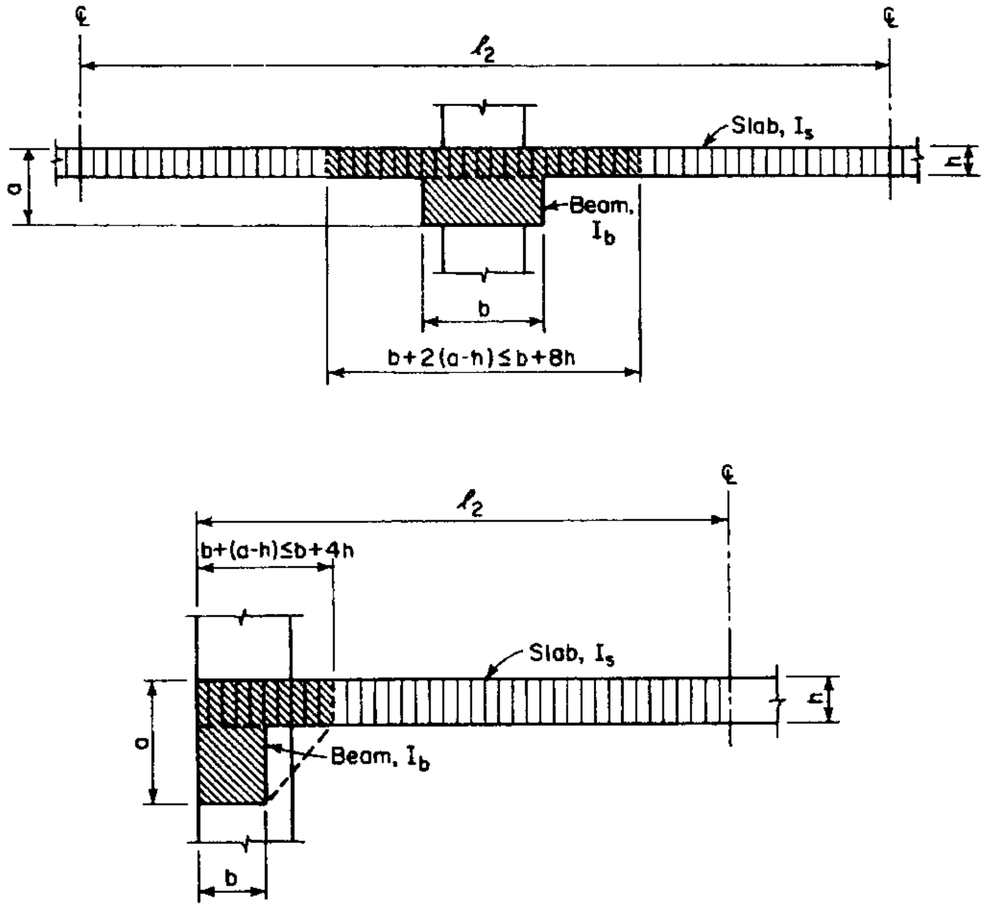


Figure 19-6 Effective Beam and Slab Sections for Computation of Stiffness Ratio α_f

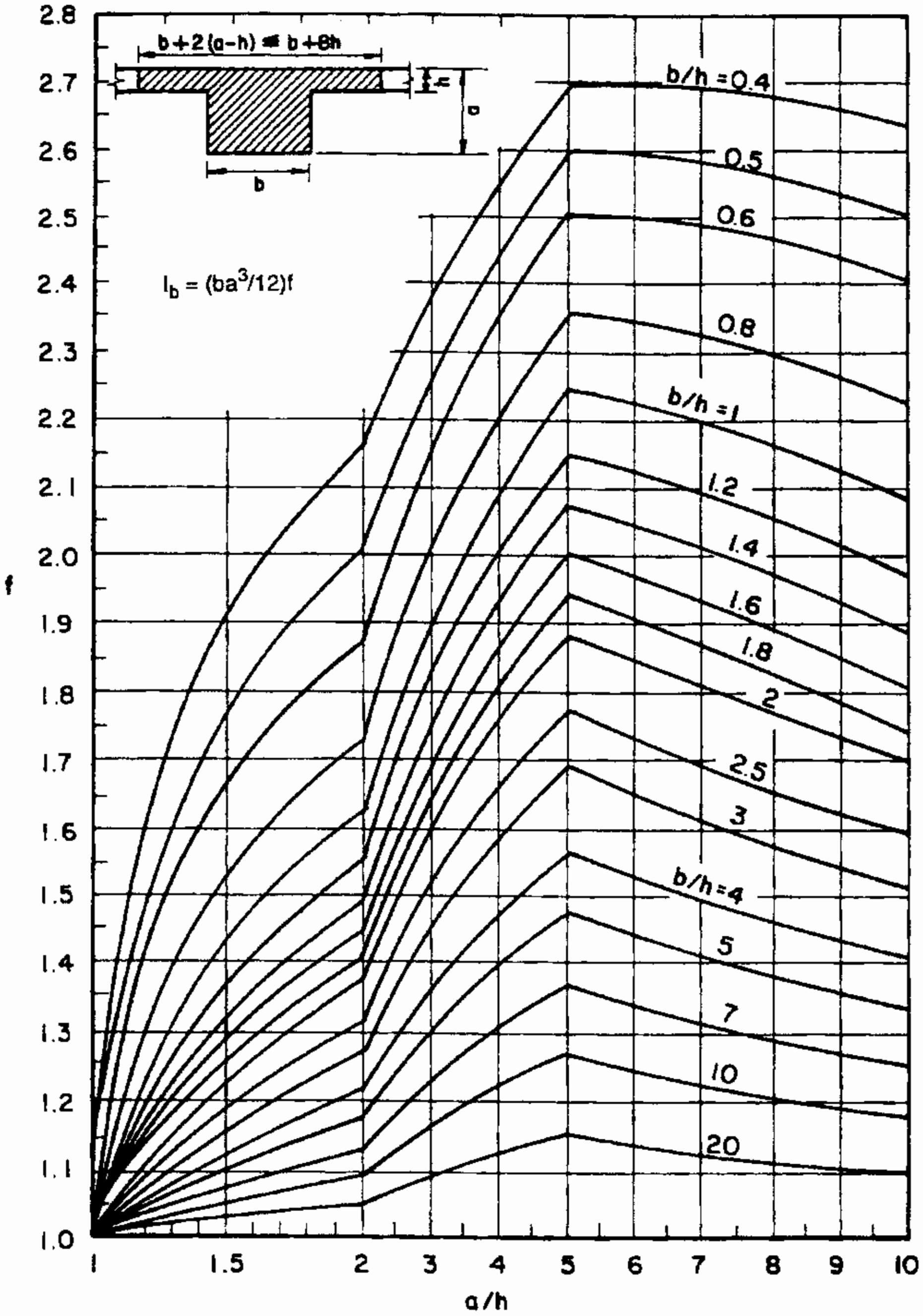


Figure 19-7 Beam Stiffness (Interior Beams)

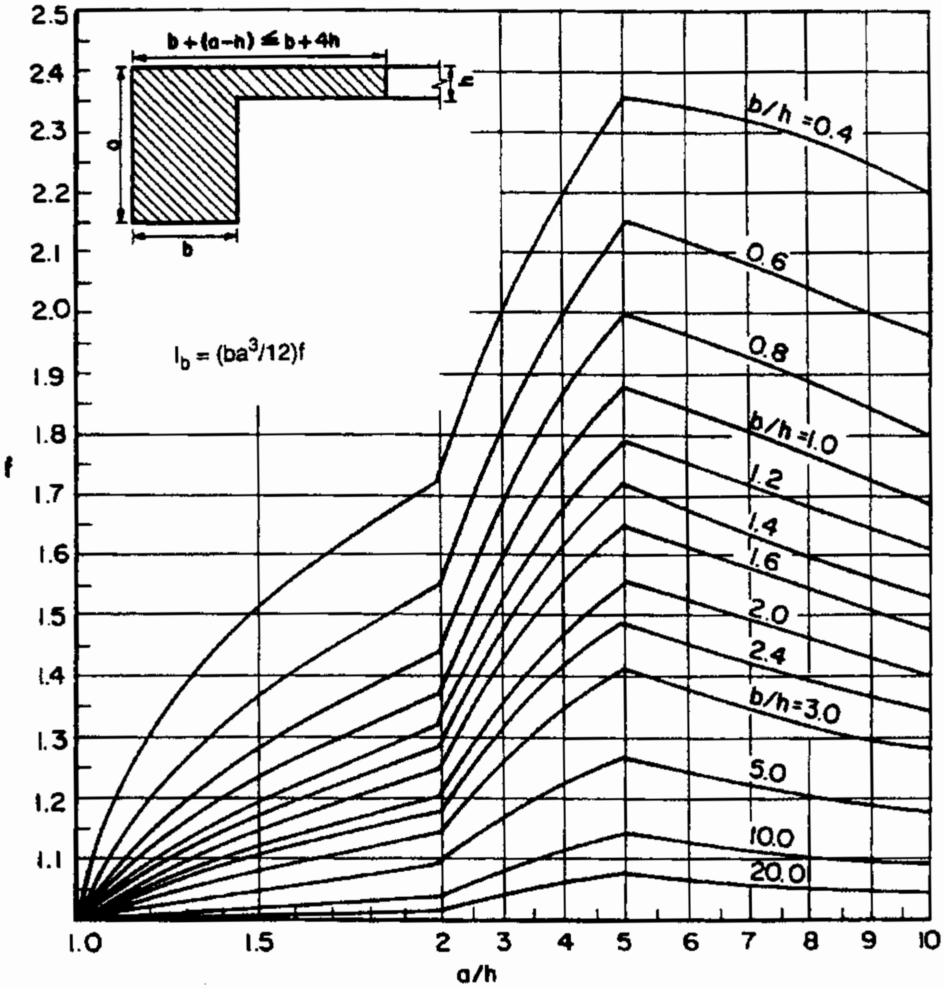
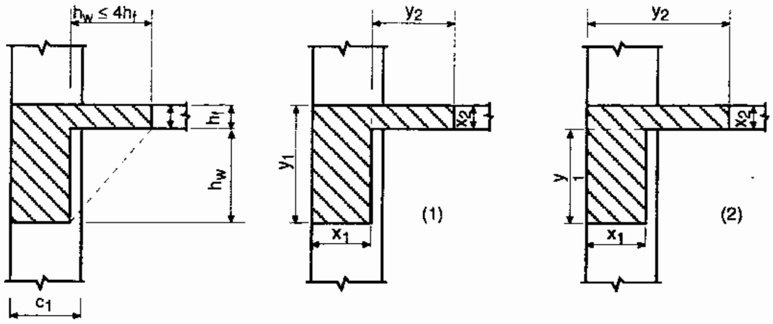


Figure 19-8 Beam Stiffness (Edge Beams)

Table 19-2 Design Aid for Computing C, Cross-Sectional Constant Defining Torsional Properties



Edge beam (ACI 13.2.4)

Use larger value of C computed from (1) or (2)

y	x*									
	4	5	6	7	8	9	10	12	14	16
12	202	369	592	868	1,118	1,538	1,900	2,557		
14	245	452	736	1,096	1,529	2,024	2,566	3,709	4,738	
16	288	534	880	1,325	1,871	2,510	3,233	4,861	6,567	8,083
18	330	619	1,024	1,554	2,212	2,996	3,900	6,013	8,397	10,813
20	373	702	1,167	1,782	2,553	3,482	4,567	7,165	10,226	13,544
22	416	785	1,312	2,011	2,895	3,968	5,233	8,317	12,055	16,275
24	458	869	1,456	2,240	3,236	4,454	5,900	9,459	13,885	19,005
27	522	994	1,672	2,583	3,748	5,183	6,900	11,197	16,628	23,101
30	586	1,119	1,888	2,926	4,260	5,912	7,900	12,925	19,373	27,197
33	650	1,243	2,104	3,269	4,772	6,641	8,900	14,653	22,117	31,293
36	714	1,369	2,320	3,612	5,284	7,370	9,900	16,381	24,860	35,389
42	842	1,619	2,752	4,298	6,308	8,828	11,900	19,837	30,349	43,581
48	970	1,869	3,183	4,984	7,332	10,286	13,900	23,293	35,836	51,773
54	1,098	2,119	3,616	5,670	8,356	11,744	15,900	26,749	41,325	59,965
60	1,226	2,369	4,048	6,356	9,380	13,202	17,900	30,205	46,813	68,157

** Small side of a rectangular cross-section with dimensions x and y.*

13.6.5 Factored Moments in Beams

When a design strip contains beams between columns, the factored moment assigned to the column strip must be distributed between the slab and the beam portions of the column strip. The amount of the column strip factored moment to be resisted by the beam varies linearly between zero and 85 percent as $\alpha_{f1}\ell_2/\ell_1$ varies between zero and 1.0. When $\alpha_{f1}\ell_2/\ell_1$ is equal to or greater than 1.0, 85 percent of the total column strip moment must be resisted by the beam. In addition, the beam section must resist the effects of loads applied directly to the beam, including weight of beam stem projecting above or below the slab.

13.6.6 Factored Moments in Middle Strips

Factored moments not assigned to the column strips must be resisted by the two half-middle strips comprising the design strip. An exception to this is a middle strip adjacent to and parallel with an edge supported by a wall, where the moment to be resisted is twice the factored moment assigned to the half middle strip corresponding to the first row of interior supports (see Fig. 19-1).

13.6.9 Factored Moments in Columns and Walls

Supporting columns and walls must resist any negative moments transferred from the slab system.

For interior columns (or walls), the approximate Eq. (13-7) may be used to determine the unbalanced moment transferred by gravity loading, unless an analysis is made considering the effects of pattern loading and unequal adjacent spans. The transfer moment is computed directly as a function of span length and gravity loading. For the more usual case with equal transverse and adjacent spans, Eq. (13-7) reduces to

$$M_u = 0.07\left(0.5q_{Lu}\ell_2\ell_n^2\right) \qquad (4)$$

where, q_{Lu} = factored live load, psf

ℓ_2 = span length transverse to ℓ_n

ℓ_n = clear span length in the direction of analysis

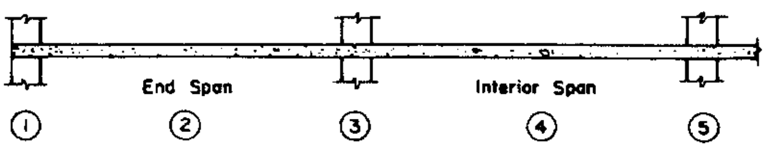
At exterior column or wall supports, the total exterior negative factored moment from the slab system (13.6.3.3) is transferred directly to the supporting members. Due to the approximate nature of the moment coefficients, it seems unwarranted to consider the change in moment from face of support to centerline of support; use the moment values from 13.6.3.3 directly.

Columns above and below the slab must resist the unbalanced support moment based on the relative column stiffnesses—generally, in proportion to column lengths above and below the slab. Again, due to the approximate nature of the moment coefficients of the Direct Design Method, the refinement of considering the change in moment from centerline of slab-beam to top or bottom of column seems unwarranted.

DESIGN AID — DIRECT DESIGN MOMENT COEFFICIENTS

Distribution of the total free-span moment M_o into negative and positive moments, and then into column and middle strip moments, involves direct application of moment coefficients to the total moment M_o. The moment coefficients are a function of location of span (interior or end), slab support conditions, and type of two-way slab system. For design convenience, moment coefficients for typical two-way slab systems are given in Tables 19-3 through 19-7. Tables 19-3 through 19-6 apply to flat plates or flat slabs with differing end support conditions. Table 19-7 applies to two-way slabs supported on beams on all four sides. Final moments for the column strip and the middle strip are directly tabulated.

Table 19-3 Design Moment Coefficients for Flat Plate or Flat Slab Supported Directly on Columns

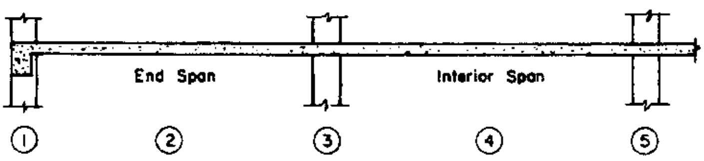


Slab Moments	End Span (1) Exterior Negative	(2) Positive	(3) First Interior Negative	Interior Span (4) Positive	(5) Interior Negative
Total Moment	$0.26M_o$	$0.52M_o$	$0.70M_o$	$0.35M_o$	$0.65M_o$
Column Strip	$0.26M_o$	$0.31M_o$	$0.53M_o$	$0.21M_o$	$0.49M_o$
Middle Strip	0	$0.21M_o$	$0.17M_o$	$0.14M_o$	$0.16M_o$

Note: All negative moments are at face of support.

The moment coefficients of Table 19-4 (flat plate with edge beams) are valid for $\beta_t \geq 2.5$. The coefficients of Table 19-7 (two-way beam-supported slabs) apply for $\alpha_{f1}\ell_2/\ell_1 \geq 1.0$ and $\beta_t \geq 2.5$. Many practical beam sizes will provide beam-to-slab stiffness ratios such that $\alpha_{f1}\ell_2/\ell_1$ and β_t will be greater than these limits, allowing moment coefficients to be taken directly from the tables, without further consideration of stiffnesses and interpolation for moment coefficients. However, if beams are present, the two stiffness parameters α_{f1} and β_t will need to be evaluated. For two-way slabs, and for $E_{cb} = E_{cs}$, the stiffness parameter α_{f1} is simply the ratio of the moments of inertia of the effective beam and slab sections in the direction of analysis, $\alpha_{f1} = I_b/I_s$, as illustrated in Fig. 19-6. Figures 19-7 and 19-8 simplify evaluation of the α_{f1} term.

Table 19-4 Design Moment Coefficients for Flat Plate or Flat Slab with Edge Beams



Slab Moments	End Span			Interior Span	
	(1)	(2)	(3)	(4)	(5)
	Exterior Negative	Positive	First Interior Negative	Positive	Interior Negative
Total Moment	$0.30M_o$	$0.50M_o$	$0.70M_o$	$0.35M_o$	$0.65M_o$
Column Strip	$0.23M_o$	$0.30M_o$	$0.53M_o$	$0.21M_o$	$0.49M_o$
Middle Strip	$0.07M_o$	$0.20M_o$	$0.17M_o$	$0.14M_o$	$0.16M_o$

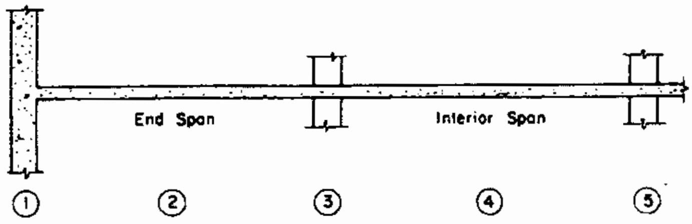
Notes: (1) All negative moments are at face of support.
(2) Torsional stiffness of edge beam is such that $\beta_t \geq 2.5$. For values of β_t less than 2.5, exterior negative column strip moment increases to $(0.30 - 0.03\,\beta_t)M_o$.

For $E_{cb} = E_{cs}$, relative stiffness provided by an edge beam is reflected by the parameter $\beta_t = C/2I_s$, where I_s is the moment of inertia of the effective slab section spanning in the direction of ℓ_1 and having a width equal to ℓ_2, i.e., $I_s = \ell_2 h^3/12$. The constant C pertains to the torsional stiffness of the effective edge beam cross-section. It is found by dividing the beam section into its component rectangles, each having a smaller dimension x and a larger dimension y, and by summing the contributions of all the parts by means of the equation:

$$C = \Sigma\left(1 - \frac{0.63x}{y}\right)\left(\frac{x^3y}{3}\right) \qquad (5)$$

The subdivision can be done in such a way as to maximize C. Table 19-2 simplifies calculation of the torsional constant C.

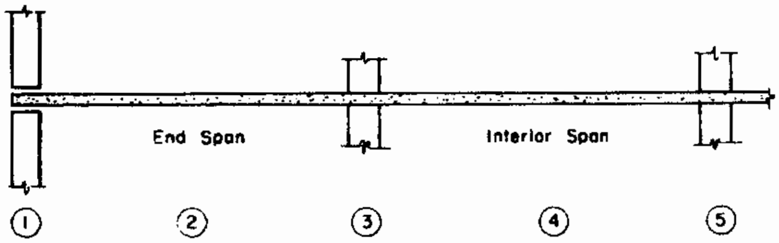
Table 19-5 Design Moment Coefficients for Flat Plate or Flat Slab with End Span Integral with Wall



Slab Moments	End Span			Interior Span	
	(1)	(2)	(3)	(4)	(5)
	Exterior Negative	Positive	First Interior Negative	Positive	Interior Negative
Total Moment	$0.65M_o$	$0.35M_o$	$0.65M_o$	$0.35M_o$	$0.65M_o$
Column Strip	$0.49M_o$	$0.21M_o$	$0.49M_o$	$0.21M_o$	$0.49M_o$
Middle Strip	$0.16M_o$	$0.14M_o$	$0.16M_o$	$0.14M_o$	$0.16M_o$

Note: All negative moments are at face of support.

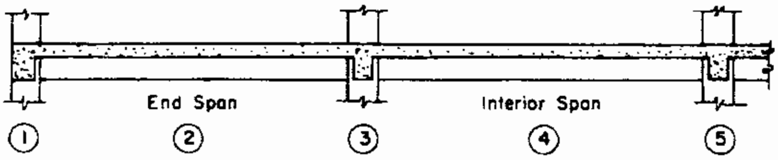
Table 19-6 Design Moment Coefficients for Flat Plate or Flat Slab with End Span Simply Supported on Wall



Slab Moments	End Span			Interior Span	
	(1)	(2)	(3)	(4)	(5)
	Exterior Negative	Positive	First Interior Negative	Positive	Interior Negative
Total Moment	0	$0.63M_o$	$0.75M_o$	$0.35M_o$	$0.65M_o$
Column Strip	0	$0.38M_o$	$0.56M_o$	$0.21M_o$	$0.49M_o$
Middle Strip	0	$0.25M_o$	$0.19M_o$	$0.14M_o$	$0.16M_o$

Note: All negative moments are at face of support.

Table 19-7 Design Moment Coefficients for Two-Way Beam-Supported Slab



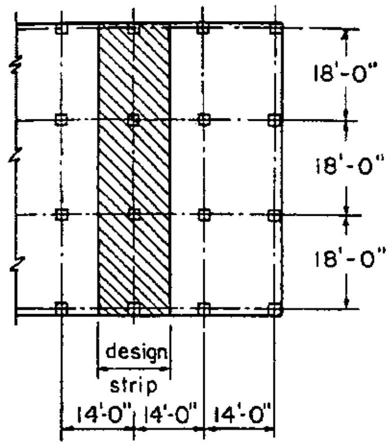
Span Ratio ℓ_2/ℓ_1	Slab and Beam Moments		End Span (1) Exterior Negative	End Span (2) Positive	End Span (3) First Interior Negative	Interior Span (4) Positive	Interior Span (5) Interior Negative
	Total Moment		$0.16M_o$	$0.57M_o$	$0.70M_o$	$0.35M_o$	$0.65M_o$
0.5	Column Strip	Beam	$0.12M_o$	$0.43M_o$	$0.54M_o$	$0.27M_o$	$0.50M_o$
		Slab	$0.02M_o$	$0.08M_o$	$0.09M_o$	$0.05M_o$	$0.09M_o$
	Middle Strip		$0.02M_o$	$0.06M_o$	$0.07M_o$	$0.03M_o$	$0.06M_o$
1.0	Column Strip	Beam	$0.10M_o$	$0.37M_o$	$0.45M_o$	$0.22M_o$	$0.42M_o$
		Slab	$0.02M_o$	$0.06M_o$	$0.08M_o$	$0.04M_o$	$0.07M_o$
	Middle Strip		$0.04M_o$	$0.14M_o$	$0.17M_o$	$0.09M_o$	$0.16M_o$
2.0	Column Strip	Beam	$0.06M_o$	$0.22M_o$	$0.27M_o$	$0.14M_o$	$0.25M_o$
		Slab	$0.01M_o$	$0.04M_o$	$0.05M_o$	$0.02M_o$	$0.04M_o$
	Middle Strip		$0.09M_o$	$0.31M_o$	$0.38M_o$	$0.19M_o$	$0.36M_o$

Notes: (1) All negative moments are at face of support.
(2) Torsional stiffness of edge beam is such that $\beta_t \geq 2.5$
(3) $\alpha_{f1}\ell_2/\ell_1 \geq 1.0$

Example 19.1—Two-Way Slab without Beams Analyzed by the Direct Design Method

Use the Direct Design Method to determine design moments for the flat plate slab system in the direction shown, for an intermediate floor.

Story height = 9 ft
Column dimensions = 16 × 16 in.
Lateral loads to be resisted by shear walls
No edge beams
Partition weight = 20 psf
Service live load = 40 psf
f'_c = 4,000 psi, normal weight concrete
f_y = 60,000 psi



Also determine the reinforcement and shear requirements at an exterior column.

Calculations and Discussion	Code Reference

1. Preliminary design for slab thickness h:

   a. Control of deflections.

   For slab systems without beams (flat plate), the minimum overall thickness h with Grade 60 reinforcement is (see Table 18-1): *9.5.3.2, Table 9.5(c)*

   $$h = \frac{\ell_n}{30} = \frac{200}{30} = 6.67 \text{ in. Use } h = 7 \text{ in.}$$

   where ℓ_n is the length of clear span in the long direction = 216 - 16 = 200 in.

   This is larger than the 5 in. minimum specified for slabs without drop panels. *9.5.3.2(a)*

   b. Shear strength of slab.

   Use an average effective depth, d ≈ 5.75 in. (3/4-in. cover and No. 4 bar)

   Factored dead load, q_{Du} = 1.2 (87.5 + 20) = 129 psf

   Factored live load, q_{Lu} = 1.6 × 40 = 64 psf

   Total factored load, q_u = 193 psf

   Investigation for wide-beam action is made on a 12-in. wide strip at a distance d from the face of support in the long direction (see Fig.19-9). *11.12.1.1*

   V_u = 0.193 × 7.854 = 1.5 kips

   $$V_c = 2\sqrt{f'_c}\, b_w d$$ *Eq. (11-3)*

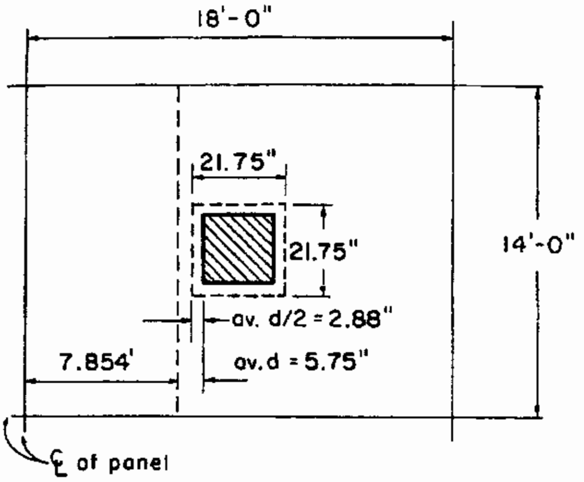


Figure 19-9 Critical Sections for One-Way and Two-Way Shear

$$= \frac{2\sqrt{4,000} \times 12 \times 5.75}{1,000} = 8.73 \text{ kips}$$

$\phi V_c = 0.75 \times 8.73 = 6.6 \text{ kips} > V_u = 1.5 \text{ kips}$ O.K.

Since there are no shear forces at the centerline of adjacent panels (see Fig. 19-9), the shear strength in two-way action at d/2 distance around a support is computed as follows:

$V_u = 0.193\ [(18 \times 14) - 1.81^2)] = 48.0 \text{ kips}$

$V_c = 4\sqrt{f'_c}\, b_o d$ (for square columns) — *Eq. (11-35)*

$$= \frac{4\sqrt{4,000} \times (4 \times 21.75) \times 5.75}{1,000} = 126.6 \text{ kips}$$

$V_u = 48.0 \text{ kips} < \phi V_c = 0.75 \times 126.6 \text{ kips} = 95.0 \text{ kips}$ O.K.

Therefore, preliminary design indicates that a 7 in. slab is adequate for control of deflection and shear strength.

2. Check applicability of Direct Design Method: — *13.6.1*

   There is a minimum of three continuous spans in each direction — *13.6.1.1*

   Long-to-short span ratio is 1.29 < 2.0 — *13.6.1.2*

Successive span lengths are equal *13.6.1.3*

Columns are not offset *13.6.1.4*

Loads are uniformly distributed with service live-to-dead load ratio of 0.37 < 2.0 *13.6.1.5*

Slab system is without beams *13.6.1.6*

3. Factored moments in slab:

a. Total factored moment per span. *13.6.2*

$$M_o = \frac{q_u \ell_2 \ell_n^2}{8}$$ *Eq. (13-4)*

$$= \frac{0.193 \times 14 \times 16.67^2}{8} = 93.6 \text{ ft-kips}$$

b. Distribution of the total factored moment M_o per span into negative and positive moments, and then into column and middle strip moments. This distribution involves direct application of the moment coefficients to the total moment M_o. Referring to Table 19-3 (flat plate without edge beams), *13.6.3* *13.6.4* *13.6.6*

	Total Moment (ft-kips)	Column Strip Moment (ft-kips)	Moment (ft-kips) in Two Half-Middle Strips*
End Span:			
Exterior Negative	$0.26M_o$ = 24.3	$0.26M_o$ = 24.3	0
Positive	$0.52M_o$ = 48.7	$0.31M_o$ = 29.0	$0.21M_o$ = 19.7
Interior Negative	$0.70M_o$ = 65.5	$0.53M_o$ = 49.6	$0.17M_o$ = 15.9
Interior Span:			
Positive	$0.35M_o$ = 32.8	$0.21M_o$ = 19.7	$0.14M_o$ = 13.1
Negative	$0.65M_o$ = 60.8	$0.49M_o$ = 45.9	$0.16M_o$ = 15.0

**That portion of the total moment M_o not resisted by the column strip is assigned to the two half-middle strips.*

Note: The factored moments may be modified by 10 percent, provided the total factored static moment in any panel is not less than that computed from Eq. (13-4). This modification is omitted here. *13.6.7*

4. Factored moments in columns: *13.6.9*

a. Interior columns, with equal spans in the direction of analysis and (different) equal spans in the transverse direction.

$$M_u = 0.07\left(0.5 q_{Lu} \ell_2 \ell_n^2\right)$$ *Eq. (13-7)*

$$= 0.07\,(0.5 \times 1.6 \times 0.04 \times 14 \times 16.67^2) = 8.7 \text{ ft-kips}$$

With the same column size and length above and below the slab,

$$M_c = \frac{8.7}{2} = 4.35 \text{ ft-kips}$$

This moment is combined with the factored axial load (for each story) for design of the interior columns.

b. Exterior columns.

Total exterior negative moment from slab must be transferred directly to the columns: $M_u = 24.3$ ft-kips. With the same column size and length above and below the slab,

$$M_c = \frac{24.3}{2} = 12.15 \text{ ft-kips}$$

This moment is combined with the factored axial load (for each story) for design of the exterior column.

5. Check slab flexural and shear strength at exterior column

a. Total flexural reinforcement required for design strip:

i. Determine reinforcement required for strip moment $M_u = 24.3$ ft-kips

Assume tension-controlled section ($\phi = 0.9$) *9.3.2*

Column strip width $b = \frac{14 \times 12}{2} = 84$ in. *13.2.1*

$$R_n = \frac{M_u}{\phi b d^2} = \frac{24.3 \times 12{,}000}{0.9 \times 84 \times 5.75^2} = 117 \text{ psi}$$

$$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right)$$

$$= \frac{0.85 \times 4}{60}\left(1 - \sqrt{1 - \frac{2 \times 117}{0.85 \times 4{,}000}}\right) = 0.0020$$

$$A_s = \rho b d = 0.0020 \times 84 \times 5.75 = 0.96 \text{ in.}^2$$

$\rho_{min} = 0.0018$ *13.3.1*

Min. $A_s = 0.0018 \times 84 \times 7 = 1.06\text{ in.}^2 > 0.96\text{ in.}^2$

Number of No. 4 bars = $\frac{1.06}{0.2} = 5.3$, say 6 bars

Maximum spacing $s_{max} = 2h = 14\text{ in.} < 18\text{ in.}$ *13.3.2*

Number of No.4 bars based on $s_{max} = \frac{84}{14} = 6$

Verify tension-controlled section:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(6 \times 0.2) \times 60}{0.85 \times 4 \times 84} = 0.25\text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{0.25}{0.85} = 0.29\text{ in.}$$

$$\varepsilon_t = \left(\frac{0.003}{c}\right) d_t - 0.003$$

$$= \left(\frac{0.003}{0.29}\right) 5.75 - 0.003 = 0.057 > 0.005$$ *10.3.4*

Therefore, section is tension-controlled.

Use 6-No. 4 bars in column strip.

ii. Check slab reinforcement at exterior column for moment transfer between slab and column

Portion of unbalanced moment transferred by flexure = $\gamma_f M_u$ *13.5.3.2*

From Fig. 16-13, Case C:

$$b_1 = c_1 + \frac{d}{2} = 16 + \frac{5.75}{2} = 18.88\text{ in.}$$

$$b_2 = c_2 + d = 16 + 5.75 = 21.75\text{ in.}$$

$$\gamma_f = \frac{1}{1 + (2/3)\sqrt{b_1/b_2}} = \frac{1}{1 + (2/3)\sqrt{18.88/21.75}} = 0.62$$ *Eq. (13-1)*

$\gamma_f M_u = 0.62 \times 24.3 = 15.1$ ft-kips

Note that the provions of 13.5.3.3 may be utilized; however, they are not in this example.

Assuming tension-controlled behavior, determine required area of reinforcement for $\gamma_f M_u = 15.1$ ft-kips:

Effective slab width $b = c_2 + 3h = 16 + 3\ (7) = 37$ in. *13.5.3.2*

$$R_n = \frac{M_u}{\phi b d^2} = \frac{15.1 \times 12{,}000}{0.9 \times 37 \times 5.75^2} = 165 \text{ psi}$$

$$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right)$$

$$= \frac{0.85 \times 4}{60}\left(1 - \sqrt{1 - \frac{2 \times 165}{0.85 \times 4000}}\right) = 0.0028$$

$$A_s = 0.0028 \times 37 \times 5.75 = 0.60 \text{ in.}^2$$

$$\text{Min. } A_s = 0.0018 \times 37 \times 7 = 0.47 \text{ in.}^2 < 0.60 \text{ in.}^2$$

$$\text{Number of No. 4 bars} = \frac{0.60}{0.2} = 3$$

Verify tension-controlled section:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(3 \times 0.2) \times 60}{0.85 \times 4 \times 37} = 0.29 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{0.29}{0.85} = 0.34 \text{ in.}$$

$$\varepsilon_t = \left(\frac{0.003}{0.34}\right) 5.75 - 0.003 = 0.048 > 0.005$$ *10.3.4*

Therefore, section is tension-controlled.

Provide the required 3-No. 4 bars by concentrating 3 of the column strip bars (6-No. 4) within the 37 in. slab width over the column. For symmetry, add one additional No. 4 bar outside of 37-in. width.

Note that the column strip section remains tension-controlled with the addition of 1-No. 4 bar.

iii. Determine reinforcement required for middle strip.

Since all of the moment at exterior columns is transferred to the column strip, provide minimum reinforcement in middle strip:

$$\text{Min. } A_s = 0.0018 \times 84 \times 7 = 1.06 \text{ in.}^2$$

$$\text{Number of No. 4 bars} = \frac{1.06}{0.2} = 5.3, \text{ say } 6$$

Maximum spacing $s_{max} = 2h = 14 \text{ in.} < 18 \text{ in.}$ *13.3..2*

$$\text{Number of No. 4 bars based on } s_{max} = \frac{84}{14} = 6$$

Provide No. 4 @ 14 in. in middle strip.

b. Check combined shear stress at inside face of critical transfer section: *11.12.6.1*

For shear strength equations, see Part 16.

$$v_u = \frac{V_u}{A_c} + \frac{\gamma_v M_u c_{AB}}{J_c}$$

Factored shear force at exterior column:

$$V_u = 0.193\left[(14 \times 9.667) - \left(\frac{18.88 \times 21.75}{144}\right)\right] = 25.6 \text{ kips}$$

When the end span moments are determined from the Direct Design Method, the fraction of unbalanced moment transferred by eccentricity of shear must be $0.3M_o = 0.3 \times 93.6 = 28.1$ ft-kips. *13.6.3.6*

$\gamma_v = 1 - \gamma_f = 1 - 0.62 = 0.38$ *Eq. (11-3)*

From Fig. 16-13, critical section propeties for edge column bending perpendicular to edge (Case C):

$$A_c = (2b_1 + b_2)\,d = [(2 \times 18.88) + 21.75] \times 5.75 = 342.2 \text{ in.}^2$$

$$\frac{J_c}{c_{AB}} = \frac{2b_1^2 d\,(b_1 + 2b_2) + d^3\,(2b_1 + b_2)}{6b_1}$$

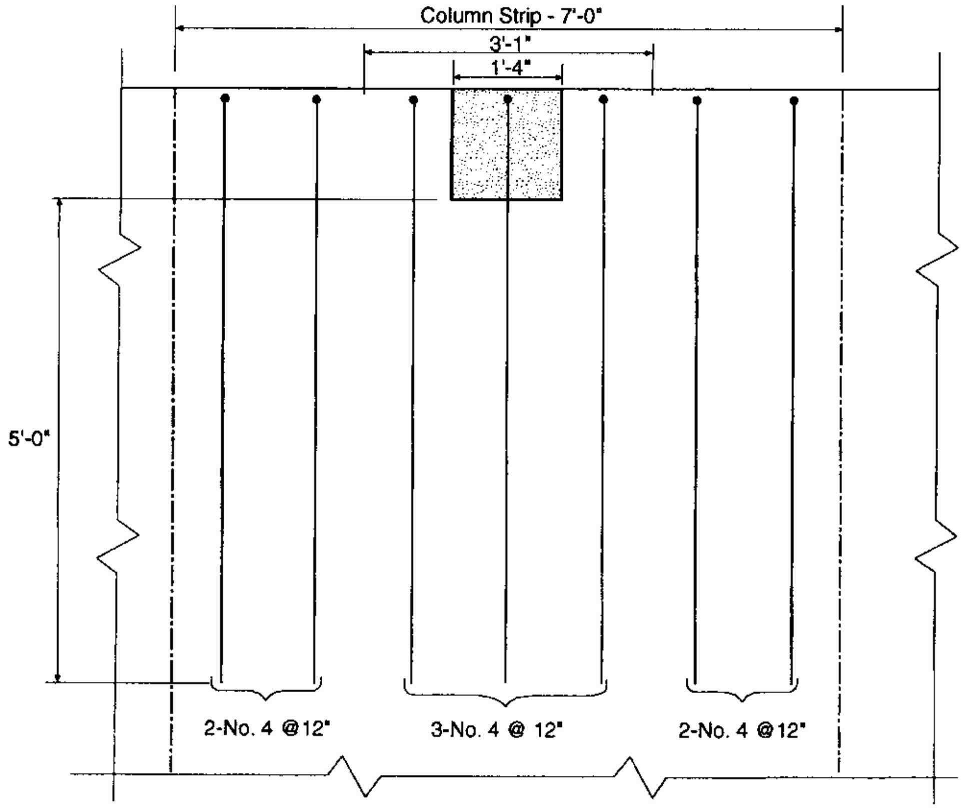
$$= \frac{2(18.88)^2(5.75)\,[18.88 + (2 \times 21.75)] + 5.75^3\,[(2 \times 18.88) + 21.75]}{6 \times 18.88}$$

$$= 2{,}357 \text{ in.}^3$$

$$v_u = \frac{25{,}600}{342.2} + \frac{0.38 \times 28.1 \times 12{,}000}{2{,}357}$$

$$= 74.8 + 54.4 = 129.2 \text{ psi}$$

Allowable shear stress $\phi v_n = \phi 4\sqrt{f'_c} = 0.75 \times 4\sqrt{4{,}000} = 189.7 \text{ psi} > v_u$ O.K. *11.12..6.2*

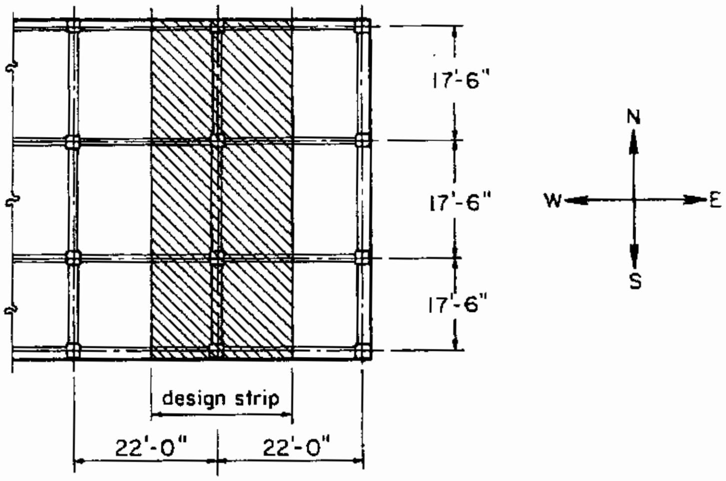


Example 19.2—Two-Way Slab with Beams Analyzed by the Direct Design Method

Use the Direct Design Method to determine design moments for the slab system in the direction shown, for an intermediate floor.

Story height = 12 ft
Edge beam dimensions = 14 × 27 in.
Interior beam dimensions = 14 × 20 in.
Column dimensions = 18 × 18 in.
Slab thickness = 6 in.
Service live load = 100 psf

f'_c = 4,000 psi (for all members),
normal weight concrete
f_y = 60,000 psi



Calculations and Discussion	Code Reference

1. Preliminary design for slab thickness h: *9.5.3*

   Control of deflections.

   With the aid of Figs. 19-6, 19-7, and 19-8, beam-to-slab flexural stiffness ratio α_f is computed as follows:

   NS edge beams:

   ℓ_2 = 141 in.

   $$\frac{a}{h} = \frac{27}{6} = 4.5$$

   $$\frac{b}{h} = \frac{14}{6} = 2.33$$

   From Fig. 19-8, f = 1.47

   $$I_b = \left(\frac{ba^3}{12}\right)f$$

   $$I_s = \frac{\ell_2 h^3}{12}$$

$$\alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{I_b}{I_s}$$ *13.0*

$$= \left(\frac{b}{\ell_2}\right)\left(\frac{a}{h}\right)^3 f$$

$$= \left(\frac{14}{141}\right)\left(\frac{27}{6}\right)^3 (1.47) = 13.30$$

EW edge beams:

$$\ell_2 = \frac{17.5 \times 12}{2} + \frac{18}{2} = 114 \text{ in.}$$

$$\alpha_f = \left(\frac{14}{114}\right)\left(\frac{27}{6}\right)^3 (1.47) = 16.45$$

NS interior beams:

ℓ_2 = 22 ft = 264 in.

$$\frac{a}{h} = \frac{20}{6} = 3.33$$

$$\frac{b}{h} = \frac{14}{6} = 2.33$$

From Fig. 19-7, f = 1.61

$$\alpha_f = \left(\frac{14}{114}\right)\left(\frac{27}{6}\right)^3 (1.47) = 16.45$$

EW interior beams:

ℓ_2 = 17.5 ft = 210 in.

$$\alpha_f = \left(\frac{14}{210}\right)\left(\frac{20}{6}\right)^3 (1.61) = 3.98$$

Since $\alpha_f > 2.0$ for all beams, Eq. (9-13) will control minimum thickness. *9.5.3.3*

Therefore,

$$h = \frac{\ell_n\left(0.8 + \frac{f_y}{200{,}000}\right)}{36 + 9\beta}$$ *Eq. (9-12)*

$$= \frac{246\left(0.8 + \frac{60{,}000}{200{,}000}\right)}{36 + 9\,(1.28)} = 5.7 \text{ in.}$$

where

$$\beta = \frac{\text{clear span in the long direction}}{\text{clear span in the short direction}} = \frac{20.5}{16} = 1.28$$

ℓ_n = clear span in long direction measured face to face of columns = 20.5 ft = 246 in.

Use 6 in. slab thickness

2. Check applicability of Direct Design Method: *13.6.1*

There is a minimum of three continuous spans in each direction *13.6.1.1*
Long-to-short span ratio is 1.26 < 2.0 *13.6.1.2*
Successive span lengths are equal *13.6.1.3*
Columns are not offset *13.6.1.4*
Loads are uniformly distributed with service live-to-dead ratio of 1.33 < 2.0 *13.6.1.5*

Check relative stiffness for slab panel: *13.6.1.6*

Interior Panel:

$\alpha_{f1} = 3.16 \qquad \ell_2 = 264$ in.

$\alpha_{f2} = 3.98 \qquad \ell_1 = 210$ in.

$$\frac{\alpha_{f1}\ell_2^2}{\alpha_{f2}\ell_1^2} = \frac{3.16 \times 264^2}{3.98 \times 210^2} = 1.25 \qquad 0.2 < 1.25 < 5.0 \quad \text{O.K.}$$ *Eq. (13-2)*

Exterior Panel:

$\alpha_{f1} = 3.16 \qquad \ell_2 = 264$ in.

$\alpha_{f2} = 16.45 \qquad \ell_1 = 210$ in.

$$\frac{\alpha_{f1}\ell_2^2}{\alpha_{f2}\ell_1^2} = \frac{3.16 \times 264^2}{16.45 \times 210^2} = 0.3 \qquad 0.2 < 0.3 < 5.0 \quad \text{O.K.}$$

Therefore, use of Direct Design Method is permitted.

3. Factored moments in slab:

Total factored moment per span — *13.6.2*

Average weight of beams stem $= \dfrac{14 \times 14}{144} \times \dfrac{150}{22} = 9.3$ psf

Weight of slab $= \dfrac{6}{12} \times 150 = 75$ psf

$w_u = 1.2(75 + 9.3) + 1.6(100) = 261$ psf — *Eq. (9-2)*

$\ell_n = 17.5 - \dfrac{18}{12} = 16$ ft

$M_o = \dfrac{q_u \ell_2 \ell_n^{\,2}}{8}$ — *Eq. (13-4)*

$= \dfrac{0.261 \times 22 \times 16^2}{8} = 183.7$ ft-kips

Distribution of moment into negative and positive moments:

Interior span: — *13.6.3.2*

Negative moment $= 0.65\,M_o = 0.65 \times 183.7 = 119.4$ ft-kips
Positive moment $= 0.35\,M_o = 0.35 \times 183.7 = 64.3$ ft-kips

End span: — *13.6.3.3*

Exterior negative $= 0.16\,M_o = 0.16 \times 183.7 = 29.4$ ft-kips
Positive $= 0.57\,M_o = 0.57 \times 183.7 = 104.7$ ft-kips
Interior negative $= 0.70\,M_o = 0.7 \times 183.7 = 128.6$ ft-kips

Note: The factored moments may be modified by 10 percent, provided the total factored static moment in any panel is not less than that computed from Eq. (13-3). This modification is omitted here. — *13.6.7*

4. Distribution of factored moments to column and middle strips: — *13.6.4*

Percentage of total negative and positive moments to column strip.

At interior support:

$$75 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right) = 75 + 30\,(1 - 1.26) = 67\% \qquad \textit{Eq. (1)}$$

where α_{f1} was computed earlier to be 3.16 (see NS interior beam above)

At exterior support:

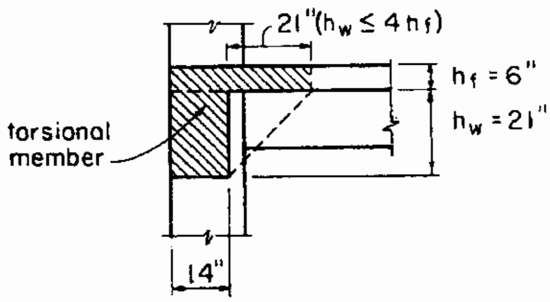
$$100 - 10\beta_t + 12\beta_t\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right) = 100 - 10\,(1.88) + 12\,(1.88)\,(1 - 1.26) = 75\% \qquad \textit{Eq. (2)}$$

where

$$\beta_t = \frac{C}{2I_s} = \frac{17{,}868}{2 \times 4752} = 1.88$$

$$I_s = \frac{\ell_2 h^3}{12} = 4{,}752 \text{ in.}^4$$

C is taken as the larger value computed (with the aid of Table 21-2) for the torsional member shown below.



x_1 = 14 in.	x_2 = 6 in.	x_1 = 14 in.	x_2 = 6 in.
y_1 = 21 in.	y_2 = 35 in.	y_1 = 27 in.	y_2 = 21 in.
C_1 = 11,141 in.^{4}	C_2 = 2248 in.^{4}	C_1 = 16,628 in.^{4}	C_2 = 1240 in.^{4}
ΣC = 11,141 + 2248 = 13,389 in.^{4}		ΣC = 16,628 + 1240 = 17,868 in.^{4}	

Positive moment:

$$60 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1.5 - \frac{\ell_2}{\ell_1}\right) = 60 + 30\,(1.5 - 1.26) = 67\% \qquad \textit{Eq. (3)}$$

Factored moments in column strips and middle strips are summarized as follows:

	Factored Moment (ft-kips)	Column Strip		Moment (ftkips) in Two Half-Middle Strips[2]
		Percent	Moment[1] (ft-kips)	
End Span:				
Exterior Negative	29.4	75	22.1	7.3
Positive	104.7	67	70.1	34.6
Interior Negative	128.6	67	86.2	42.4
Interior Span:				
Negative	119.4	67	80.0	39.4
Positive	64.3	67	43.1	21.2

[1] *Since $\alpha_1\ell_2/\ell_1 > 1.0$, beams must be proportioned to resist 85 percent of column strip moment per 13.6.5.1.*

[2] *That portion of the factored moment not resisted by the column strip is assigned to the half-middle strips.*

5. Factored moments in columns: *13.6.9*

a. Interior columns, with equal spans in the direction of analysis and (different) equal spans in the transverse direction. *13.6.9*

$$M_u = 0.07\left(0.5q_{Lu}\ell_2\ell_n^{\,2}\right)$$ *Eq. (13-7)*

$$= 0.07\,(0.5 \times 1.6 \times 0.1 \times 22 \times 16^2) = 31.5 \text{ ft-kips}$$

With the same column size and length above and below the slab,

$$M_c = \frac{31.5}{2} = 15.8 \text{ ft-kips}$$

This moment is combined with the factored axial load (for each story) for design of the interior columns.

b. Exterior columns.

The total exterior negative moment from the slab beam is transferred to the exterior columns; with the same column size and length above and below the slab system:

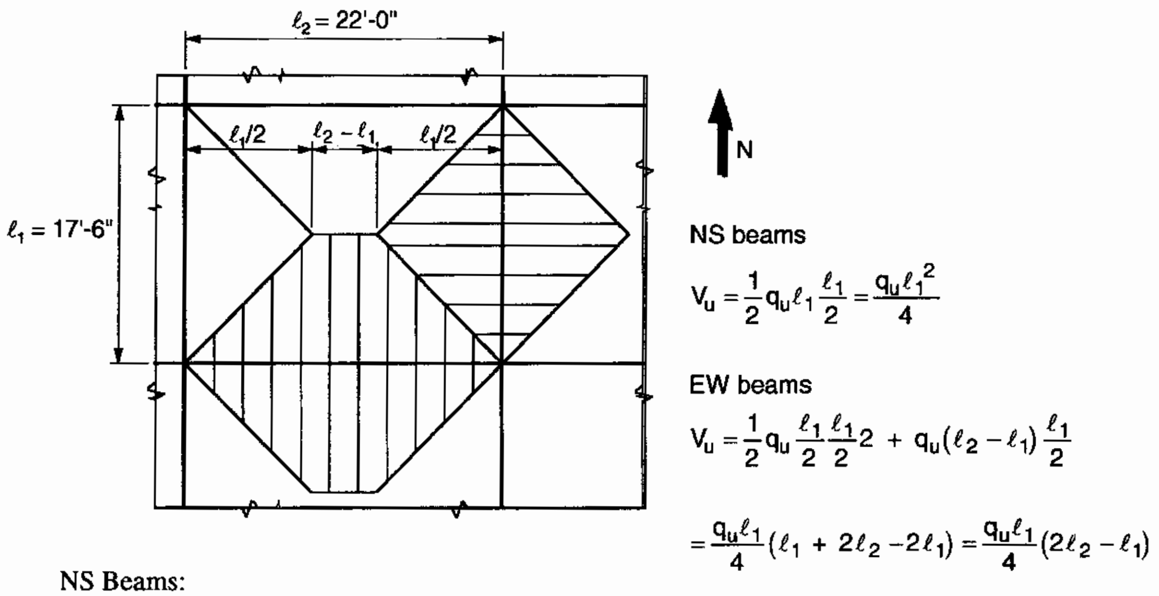
$$M_c = \frac{29.4}{2} = 14.7 \text{ ft-kips}$$

6. Shear strength:

   a. Beams.

Since $\alpha_{f1}\ell_2/\ell_1 > 1$ for all beams, they must resist total shear (b_w = 14 in., d = 17 in.). *13.6.8.1*

Only interior beams will be checked here, because they carry much higher shear forces than the edge beams.



NS Beams:

$$V_u = \frac{q_u \ell_1^2}{4} = \frac{0.261\,(17.5)^2}{4} = 20.0 \text{ kips}$$

$$\phi V_c = \phi 2\sqrt{f'_c}\, b_w d$$ *Eq. (11-3)*

$$= 0.75 \times 2\sqrt{4{,}000} \times 14 \times 17/1{,}000 = 22.6 \text{ kips} > V_u$$

Provide minimum shear reinforcement per 11.5.5.3. *11.5.5.1*

EW Beams:

$$V_u = \frac{q_u \ell_1 (2\ell_2 - \ell_1)}{4}$$

$$= \frac{0.261 \times 17.5\left[(2 \times 22) - 17.5\right]}{4} = 30.3 \text{ kips} > \phi V_c = 22.6 \text{ kips N.G.}$$

Required shear strength to be provided by shear reinforcement:

$$V_s = (V_u - \phi V_c)/\phi = (30.3 - 22.6)/0.75 = 10.3 \text{ kips}$$

b. Slabs ($b_w = 12$ in., $d = 5$ in.). *13.6.8.4*

$$q_u = (1.2 \times 75) + (1.6 \times 100) = 250 \text{ psf}$$

$$V_u = \frac{q_u \ell_1}{2} = \frac{0.25 \times 17.5}{2} = 2.2 \text{ kips}$$

$$\phi V_c = \phi 2\sqrt{f'_c}\, b_w d$$

$$= 0.75 \times 2\sqrt{4{,}000} \times 12 \times 5/1{,}000 = 5.7 \text{ kips} > V_u = 2.2 \text{ kips O.K.}$$

Shear strength of slab is adequate without shear reinforcement.

7. Edge beams must be designed to resist moment not transferred to exterior columns by interior beams, in accordance with 11.6.

20

Two-Way Slabs— Equivalent Frame Method

GENERAL CONSIDERATIONS

The Equivalent Frame Method of analysis converts a three-dimensional frame system with two-way slabs into a series of two-dimensional frames (slab-beams and columns), with each frame extending the full height of the building, as illustrated in Fig. 20-1. The width of each equivalent frame extends to mid-span between column centerlines. The complete analysis of the two-way slab system for a building consists of analyzing a series of equivalent interior and exterior frames spanning longitudinally and transversely through the building. For gravity loading, the slab-beams at each floor or roof (level) may be analyzed separately, with the far ends of attached columns considered fixed (13.7.2.5).

The Equivalent Frame Method of elastic analysis applies to buildings with columns laid out on a basically orthogonal grid, with column lines extending longitudinally and transversely through the building. The analysis method is applicable to slabs with or without beams between supports.

The Equivalent Frame Method may be used for lateral load analysis if the stiffnesses of frame members are modified to account for cracking and other relevant factors. See discussion on 13.5.1.2 in Part 18.

PRELIMINARY DESIGN

Before proceeding with Equivalent Frame analysis, a preliminary slab thickness h needs to be determined for control of deflections, according to the minimum thickness requirements of 9.5.3. Table 18-1 and Fig. 18-3 may be used to simplify minimum thickness computations. For slab systems without beams, it is advisable at this stage of design to check the shear strength of the slab in the vicinity of columns or other support locations, according to the special provisions for slabs of 11.12. See discussion on 13.5.4 in Part 18.

13.7.2 Equivalent Frame

Application of the frame definitions given in 13.7.2, 13.2.1, and 13.2.2 is illustrated in Figs. 20-1 and 20-2. Some judgment is required in applying the definitions given in 13.2.1 for slab systems with varying span lengths along the design strip. Members of the equivalent frame are slab-beams and torsional members (transverse horizontal members) supported by columns (vertical members). The torsional members provide moment transfer between the slab-beams and the columns. The equivalent frame members are illustrated in Fig. 20-3. The initial step in the frame analysis requires that the flexural stiffness of the equivalent frame members be determined.

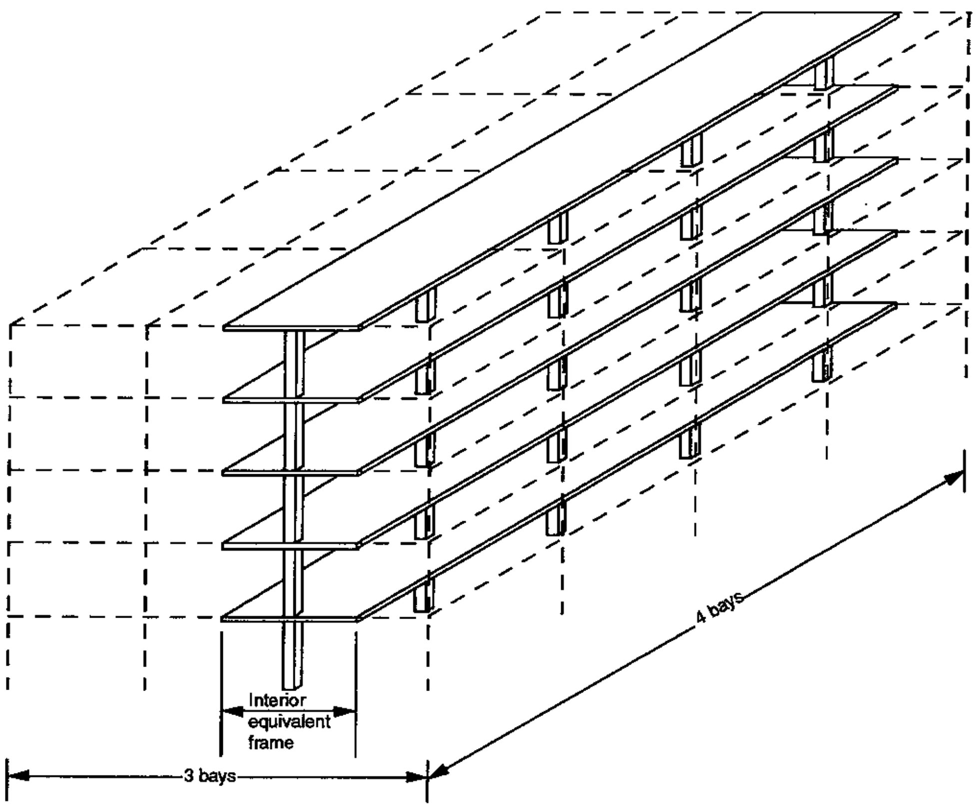


Figure 20-1 Equivalent Frames for 5-Story Building

13.7.3 Slab-Beams

Common types of slab systems with and without beams between supports are illustrated in Figs. 20-4 and 20-5. Cross-sections for determining the stiffness of the slab-beam members K_{sb} between support centerlines are shown for each type. The equivalent slab-beam stiffness diagrams may be used to determine moment distribution constants and fixed-end moments for Equivalent Frame analysis.

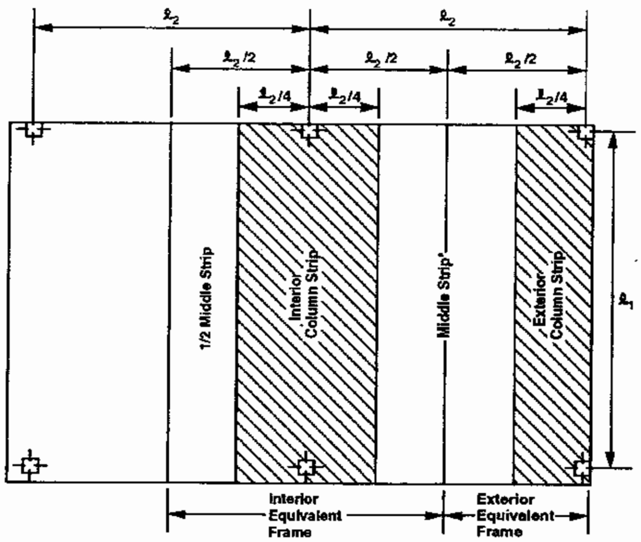
Stiffness calculations are based on the following considerations:

a. The moment of inertia of the slab-beam between faces of supports is based on the gross cross-sectional area of the concrete. Variation in the moment of inertia along the axis of the slab-beam between supports must be taken into account (13.7.3.2).

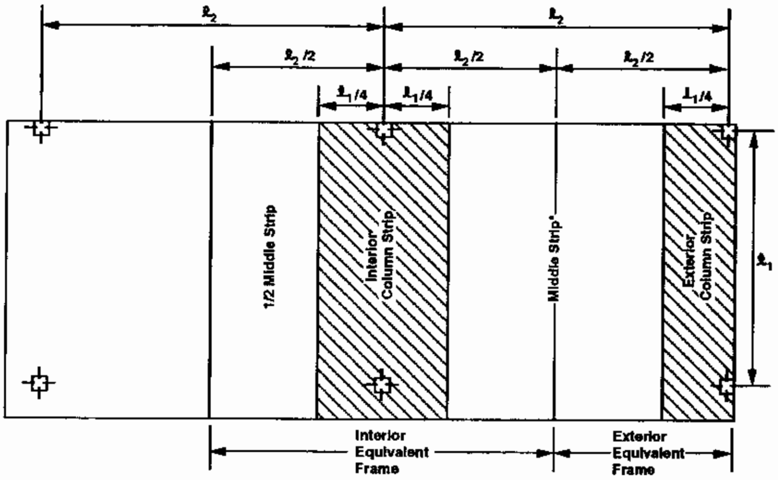
b. A support is defined as a column, capital, bracket or wall. Note that a beam is not considered a supporting member for the equivalent frame (R13.7.3.3).

c. The moment of inertia of the slab-beam from the face of support to the centerline of support is assumed equal to the moment of inertia of the slab-beam at the face of support, divided by the quantity $(1 - c_2/\ell_2)^2$ (13.7.3.3).

The magnification factor $1/(1 - c_2/\ell_2)^2$ applied to the moment of inertia between support face and support centerline, in effect, makes each slab-beam at least a haunched member within its length. Consequently, stiffness and carryover factors and fixed-end moments based on the usual assumptions of uniform prismatic members cannot be applied to the slab-beam members.



(a) Design strip for $\ell_2 \le \ell_1$



(b) Design strip for $\ell_2 \ge \ell_1$

*When edge of exterior design strip is supported by a wall, the factored moment resisted by this middle strip is defined in 13.6.6.3.

Figure 20-2 Design Strips of Equivalent Frame

Tables A1 through A6 in Appendix 20A at the end of this chapter give stiffness coefficients, carry-over factors, and fixed-end moment (at left support) coefficients for different geometric and loading configurations. A wide range of column size-to-span ratios in both longitudinal and transverse directions is covered in the tables. Table A1 can be used for flat plates and two-way slabs with beams. Tables A2 through A5 are intended to be used for flat slabs and waffle slabs with various drop (solid head) depths. Table A6 covers the unusual case of a flat plate combined with a flat slab. Fixed-end moment coefficients are provided for both uniform and partially uniform loads. Partial load coefficients were developed for loads distributed over a length of span equal to $0.2\ell_1$. However, loads acting over longer portions of span may be considered by summing the effects of loads acting over each $0.2\ell_1$ interval. For example, if the partial loading extends over $0.6\ell_1$, then the coefficients corresponding to three consecutive $0.2\ell_1$ intervals are to be added. This provides flexibility in the arrangement of loading. For concentrated loads, a high intensity of partial loading may be considered at the appropriate location, and assumed to be distributed over $0.2\ell_1$. For parameter values in between those listed, interpolation may be made. Stiffness diagrams are shown on each table. With appropriate engineering judgment, different span conditions may be considered with the help of information given in these tables.

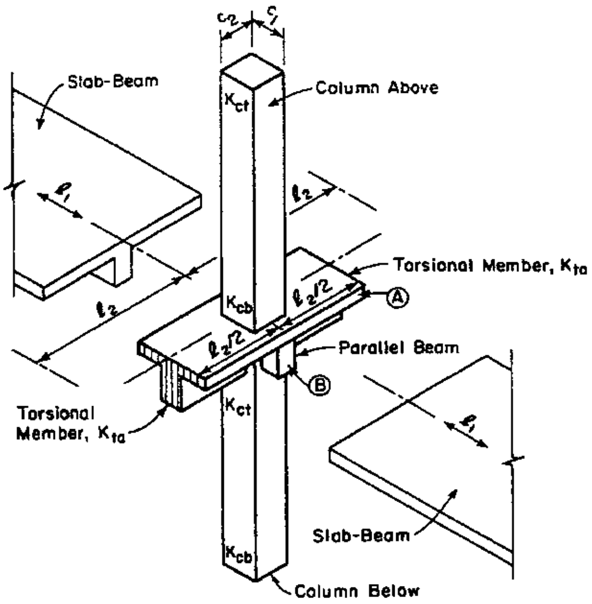


Figure 20-3 Equivalent Frame Members

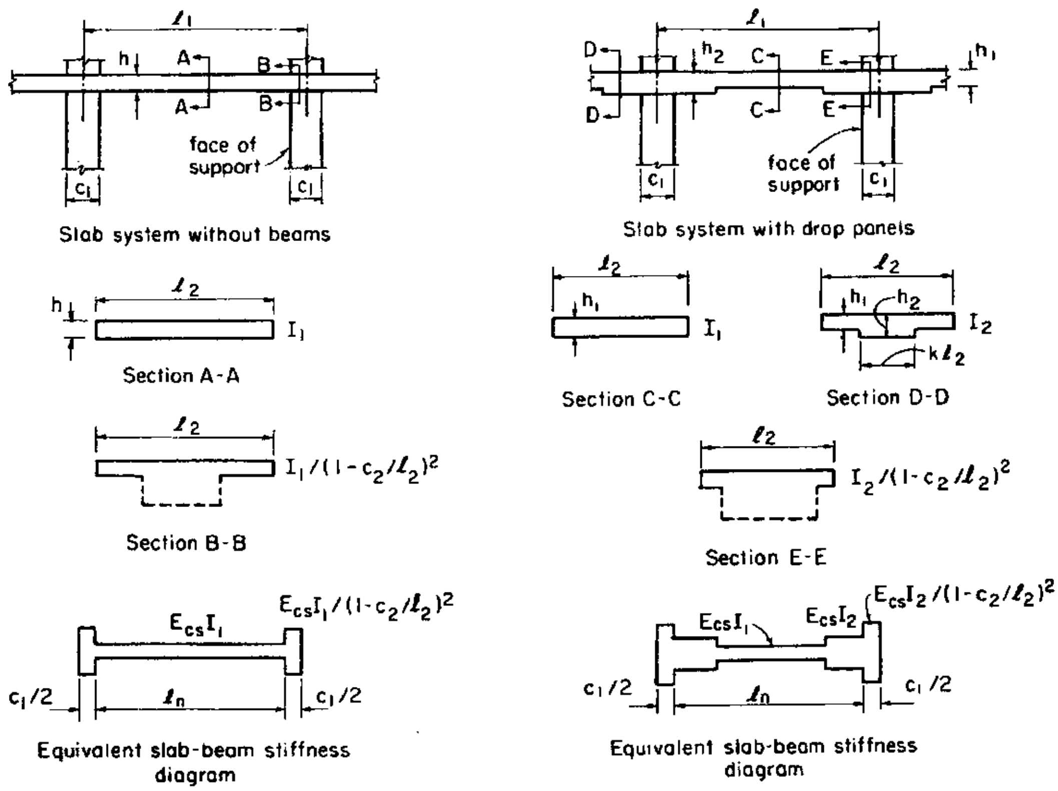


Figure 20-4 Sections for Calculating Slab-Beam Stiffness K_{sb}

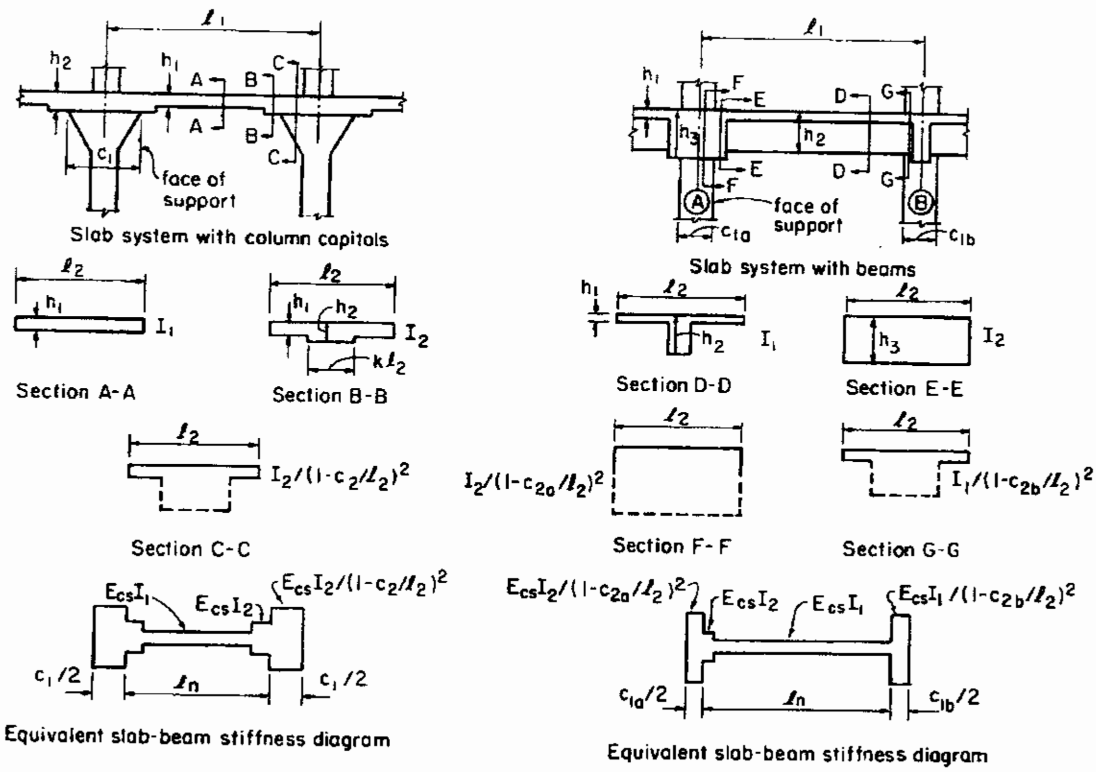


Figure 20-5 Sections for Calculating Slab-Beam Stiffness K_{sb}

13.7.4 Columns

Common types of column end support conditions for slab systems are illustrated in Fig. 20-6. The column stiffness is based on a height of column ℓ_c measured from the mid-depth of the slab above to the mid-depth of the slab below. The column stiffness diagrams may be used to determine column flexural stiffness, K_c. The stiffness diagrams are based on the following considerations:

a. The moment of inertia of the column outside the slab-beam joint is based on the gross cross-sectional area of the concrete. Variation in the moment of inertia along the axis of the column between slab-beam joints is taken into account. For columns with capitals, the moment of inertia is assumed to vary linearly from the base of the capital to the bottom of the slab-beam (13.7.4.1 and 13.7.4.2).

b. The moment of inertia is assumed infinite ($I = \infty$) from the top to the bottom of the slab-beam at the joint. As with the slab-beam members, the stiffness factor K_c for the columns cannot be based on the assumption of uniform prismatic members (13.7.4.3).

Table A7 in Appendix 20A can be used to determine the actual column stiffnesses and carry-over factors.

13.7.5 Torsional Members

Torsional members for common slab-beam joints are illustrated in Fig. 20-7. The cross-section of a torsional member is the largest of those defined by the three conditions given in 13.7.5.1. The governing condition (a), (b), or (c) is indicated below each illustration in Fig. 20-7.

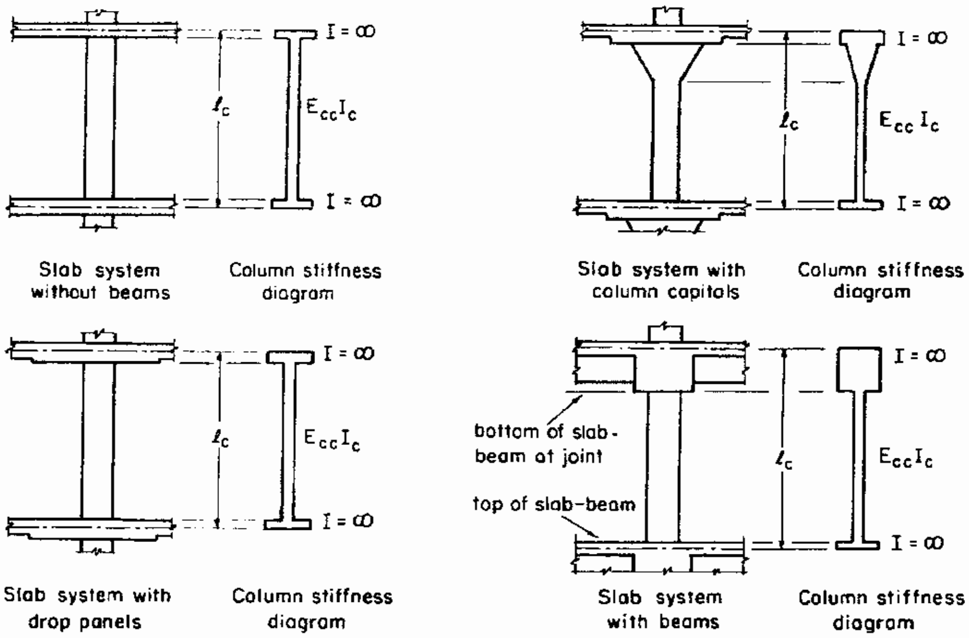


Figure 20-6 Sections for Calculating Column Stiffness K_c

The stiffness K_t of the torsional member is calculated by the following expression:

$$K_t = \Sigma\left[\frac{9E_{cs}C}{\ell_2\left[1 - (c_2/\ell_2)\right]^3}\right] \tag{1}$$

where the summation extends over torsional members framing into a joint: two for interior frames, and one for exterior frames.

The term C is a cross-sectional constant that defines the torsional properties of each torsional member framing into a joint:

$$C = \Sigma\left[1 - 0.63\left(\frac{x}{y}\right)\right]\frac{x^3y}{3} \tag{2}$$

where x is the shorter dimension of a rectangular part and y is the longer dimension of a rectangular part.

The value of C is computed by dividing the cross section of a torsional member into separate rectangular parts and summing the C values for the component rectangles. It is appropriate to subdivide the cross section in a manner that results in the largest possible value of C. Application of the C expression is illustrated in Fig. 20-8.

If beams frame into the support in the direction moments are being determined, the torsional stiffness K_t given by Eq. (1) needs to be increased as follows:

$$K_{ta} = \frac{K_t I_{sb}}{I_s}$$

where K_{ta} = increased torsional stiffness due to the parallel beam (note parallel beam shown in Fig. 20-3)

I_s = moment of inertia of a width of slab equal to the full width between panel centerlines, ℓ_2, excluding that portion of the beam stem extending above and below the slab (note part A in Fig. 20-3).

$$= \frac{\ell_2 h^3}{12}$$

I_{sb} = moment of inertia of the slab section specified for I_s including that portion of the beam stem extending above and below the slab (for the parallel beam illustrated in Fig. 20-3, I_{sb} is for the full tee section shown).

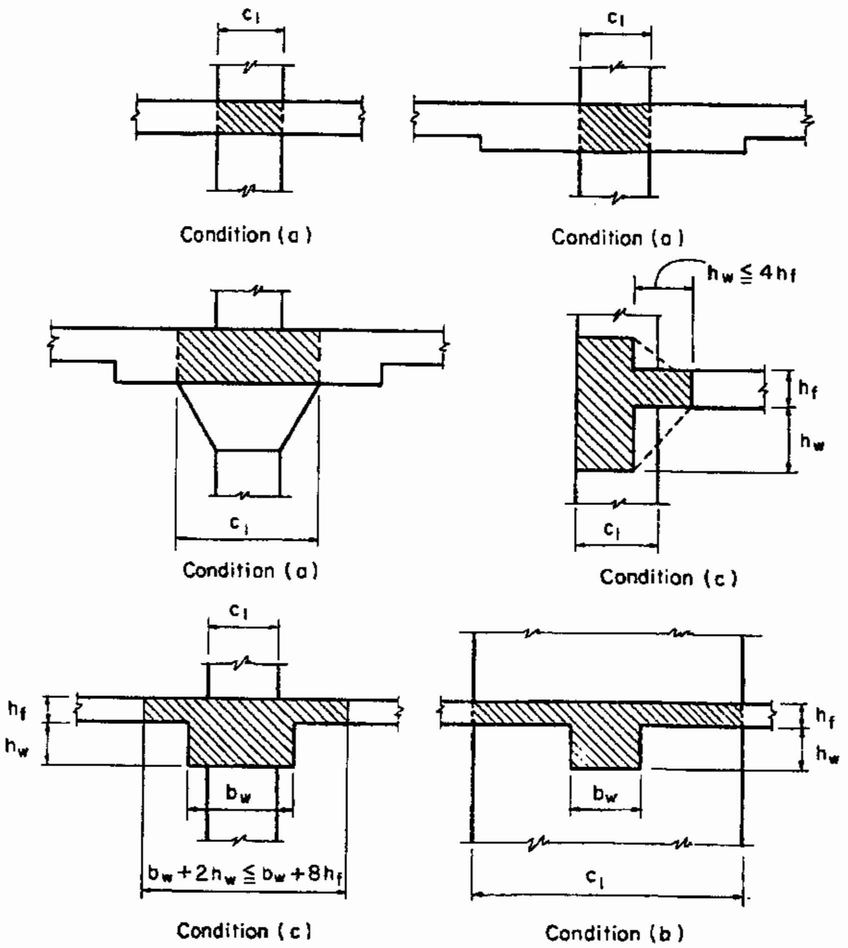


Figure 20-7 Torsional Members

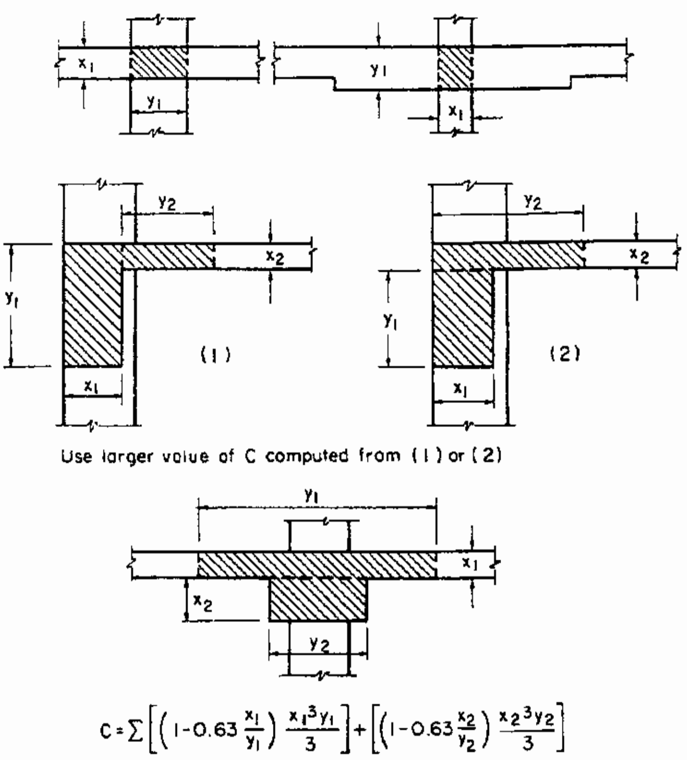


Figure 20-8 Cross-Sectional Constant C, Defining Torsional Properties of a Torsional Member

Equivalent Columns (R13.7.4)

With the publication of ACI 318-83, the equivalent column concept of defining a single-stiffness element consisting of the actual columns above and below the slab-beams plus an attached transverse torsional member was eliminated from the code. With the increasing use of computers for two-way slab analysis by the Equivalent Frame Method, the concept of combining stiffnesses of actual columns and torsional members into a single stiffness has lost much of it attractiveness. The equivalent column was, however, retained in the commentary until the 1989 edition of the code, as an aid to analysis where slab-beams at different floor levels are analyzed separately for gravity loads, especially when using moment distribution or other hand calculation procedures for the analysis. While the equivalent column concept is still recognized by R13.7.4, the detailed procedure contained in the commentary since the '83 edition for calculating the equivalent column stiffness, K_{ec}, was deleted from R13.7.5 of the '95 code.

Both Examples 20.1 and 20.2 utilize the equivalent column concept with moment distribution for gravity load analysis.

The equivalent column concept modifies the column stiffness to account for the torsional flexibility of the slab-to-column connection which reduces its efficiency for transmission of moments. An equivalent column is illustrated in Fig. 20-3. The equivalent column consists of the actual columns above and below the slab-beams, plus "attached" torsional members on both sides of the columns, extending to the centerlines of the adjacent panels. Note that for an edge frame, the attached torsional member is on one side only. The presence of parallel beams will also influence the stiffness of the equivalent column.

The flexural stiffness of the equivalent column K_{ec} is given in terms of its inverse, or flexibility, as follows:

$$\frac{1}{K_{ec}} = \frac{1}{\Sigma K_c} + \frac{1}{\Sigma K_t}$$

For computational purposes, the designer may prefer that the above expression be given directly in terms of stiffness as follows:

$$K_{ec} = \frac{\Sigma K_c \times \Sigma K_t}{\Sigma K_c + \Sigma K_t}$$

Stiffnesses of the actual columns, K_c, and torsional members, K_t must comply with 13.7.4 and 13.7.5.

After the values of K_c and K_t are determined, the equivalent column stiffness K_{ec} is computed. Using Fig. 20-3 for illustration,

$$K_{ec} = \frac{(K_{ct} + K_{cb})\,(K_{ta} + K_{ta})}{K_{ct} + K_{cb} + K_{ta} + K_{ta}}$$

where K_{ct} = flexural stiffness at top of lower column framing into joint,

K_{cb} = flexural stiffness at bottom of upper column framing into joint,

K_{ta} = torsional stiffness of each torsional member, one on each side of the column, increased due to the parallel beam (if any).

13.7.6 Arrangement of Live Load

In the usual case where the exact loading pattern is not known, the maximum factored moments are developed with loading conditions illustrated by the three-span partial frame in Fig. 20-9, and described as follows:

a. When the service live load does not exceed three-quarters of the service dead load, only loading pattern (1) with full factored live load on all spans need be analyzed for negative and positive factored moments.

b. When the service live-to-dead load ratio exceeds three-quarters, the five loading patterns shown need to be analyzed to determine all factored moments in the slab-beam members. Loading patterns (2) through (5) consider partial factored live loads for determining factored moments. However, with partial live loading, the factored moments cannot be taken less than those occurring with full factored live load on all spans; hence load pattern (1) needs to be included in the analysis.

For slab systems with beams, loads supported directly by the beams (such as the weight of the beam stem or a wall supported directly by the beams) may be inconvenient to include in the frame analysis for the slab loads, $w_d + w_\ell$. An additional frame analysis may be required with the beam section designed to carry these loads in addition to the portion of the slab moments assigned to the beams.

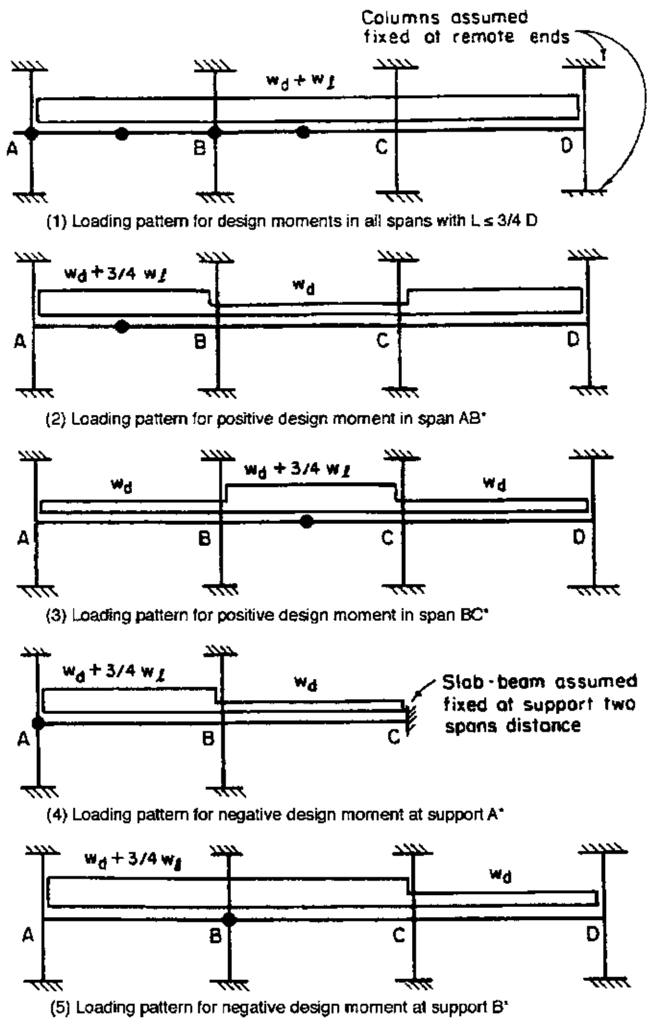


Figure 20-9 Partial Frame Analysis for Vertical Loading

13.7.7 Factored Moments

Moment distribution is probably the most convenient hand calculation method for analyzing partial frames involving several continuous spans with the far ends of upper and lower columns fixed. The mechanics of the method will not be described here, except for a brief discussion of the following two points: (1) the use of the equivalent column concept to determine joint distribution factors and (2) the proper procedure to distribute the equivalent column moment obtained in the frame analysis to the actual columns above and below the slab-beam joint. See Examples 20.1 and 20.2.

A frame joint with stiffness factors K shown for each member framing into the joint is illustrated in Fig. 20-10. Expressions are given below for the moment distribution factors DF at the joint, using the equivalent column stiffness, K_{ec}. These distribution factors are used directly in the moment distribution procedure.

Equivalent column stiffness,

$$K_{ec} = \frac{\Sigma K_c \times \Sigma K_t}{\Sigma K_c + \Sigma K_t}$$

$$= \frac{(K_{ct} + K_{cb})(K_t + K_t)}{K_{ct} + K_{cb} + K_t + K_t}$$

Slab-beam distribution factor,

$$\text{DF (span 2-1)} = \frac{K_{b1}}{K_{b1} + K_{b2} + K_{ec}}$$

$$\text{DF (span 2-3)} = \frac{K_{b2}}{K_{b1} + K_{b2} + K_{ec}}$$

Equivalent column distribution factor (unbalanced moment from slab-beam),

$$\text{DF} = \frac{K_{ec}}{K_{b1} + K_{b2} + K_{ec}}$$

The unbalanced moment determined for the equivalent column in the moment distribution cycles is distributed to the actual columns above and below the slab-beam in proportion to the actual column stiffnesses at the joint. Referring to Fig. 20-10:

$$\text{Portion of unbalanced moment to upper column} = \frac{K_{cb}}{(K_{cb} + K_{ct})}$$

$$\text{Portion of unbalanced moment to lower column} = \frac{K_{ct}}{(K_{cb} + K_{ct})}$$

The "actual" columns are then designed for these moments.

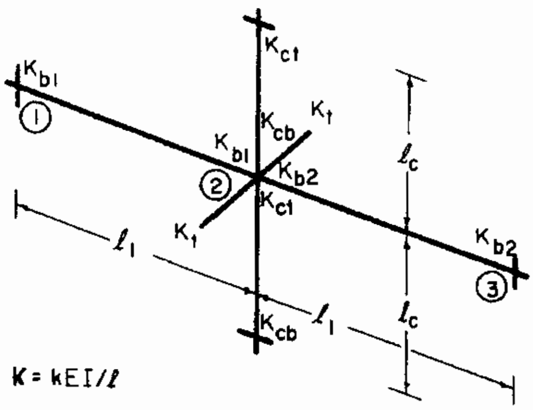


Figure 20-10 Moment Distribution Factors DF

13.7.7.1 - 13.7.7.3 Negative Factored Moments—Negative factored moments for design must be taken at faces of rectilinear supports, but not at a distance greater than $0.175\ell_1$ from the center of a support. This absolute value is a limit on long narrow supports in order to prevent undue reduction in design moment. The support member is defined as a column, capital, bracket or wall. Non-rectangular supports should be treated as square supports having the same cross-sectional area. Note that for slab systems with beams, the faces of beams are not considered face-of-support locations. Locations of the critical section for negative factored moment for various support conditions are illustrated in Fig. 20-11. Note the special requirements illustrated for exterior supports.

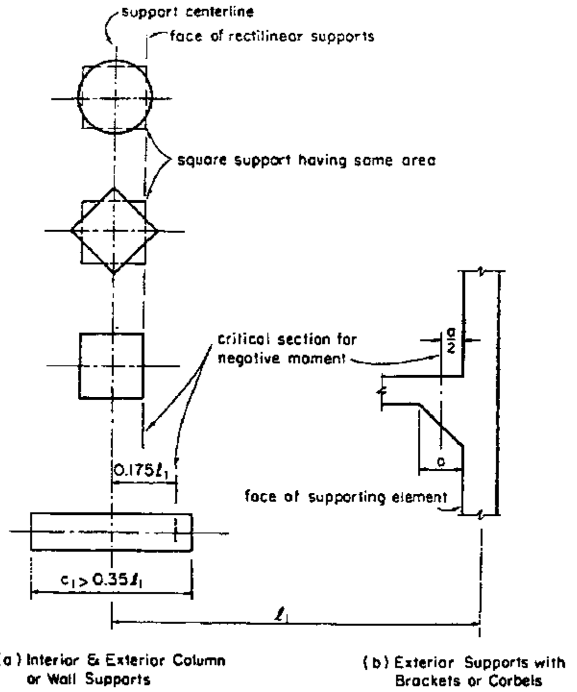


Figure 20-11 Critical Sections for Negative Factored Moment

13.7.7.4 Moment Redistribution—Should a designer choose to use the Equivalent Frame Method to analyze a slab system that meets the limitations of the Direct Design Method, the factored moments may be reduced so that the total static factored moment (sum of the average negative and positive moments) need not exceed M_0 computed by Eq. (13-4). This permissible reduction is illustrated in Fig. 20-12.

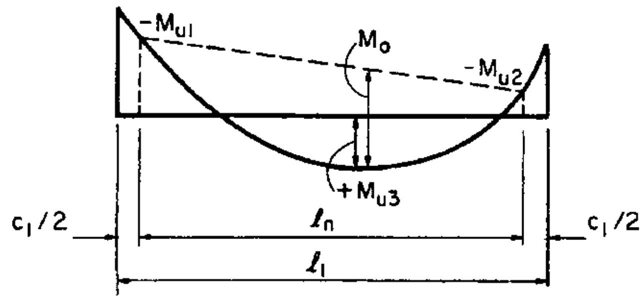


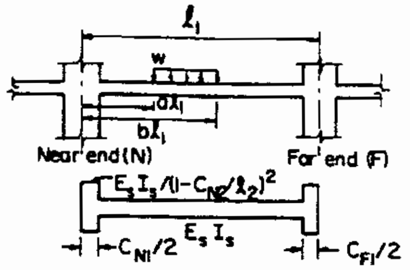
Figure 20-12 Total Static Design Moment for a Span

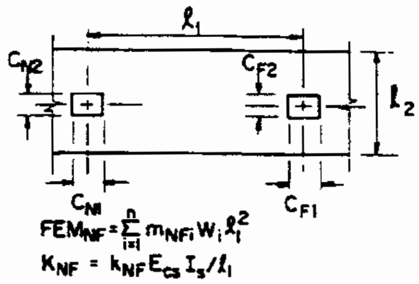
Since the Equivalent Frame Method of analysis is not an approximate method, the moment redistribution allowed in 8.4 may be used. Excessive cracking may result if these provisions are imprudently applied. The burden of judgment is left to the designer as to what, if any, redistribution is warranted.

13.7.7.5 Factored Moments in Column Strips and Middle Strips—Negative and positive factored moments may be distributed to the column strip and the two half-middle strips of the slab-beam in accordance with 13.6.4, 13.6.5 and 13.6.6, provided that the requirement of 13.6.1.6 is satisfied. See discussion on 13.6.4, 13.6.5, 13.6.6 in Part 19.

APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS

Table A1 Moment Distribution Constants for Slab-Beam Members

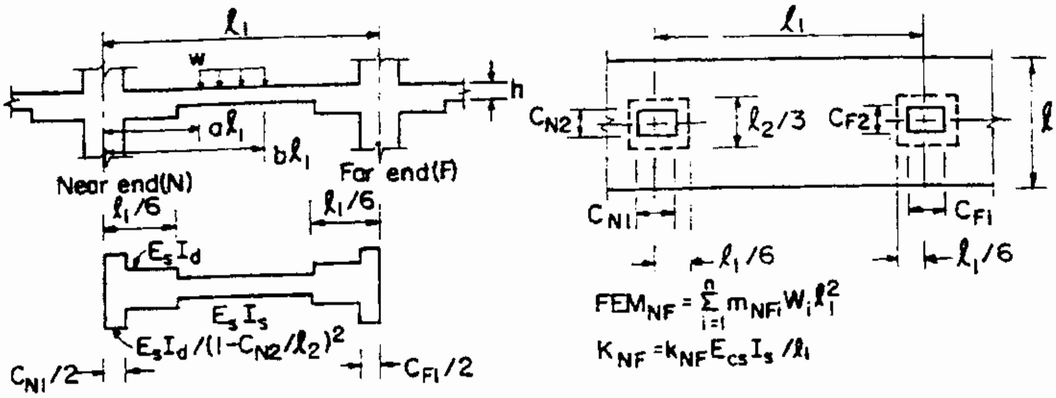




C_{N1}/ℓ_1	C_{N2}/ℓ_2	Stiffness Factors k_{NF}	Carry Over Factors C_{NF}	Unif. Load Fixed end M. Coeff. (m_{NF})	Fixed end moment Coeff. (m_{NF}) for (b−a) = 0.2				
					a = 0.0	a = 0.2	a = 0.4	a = 0.6	a = 0.8
					$C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$				
0.00	—	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.00226
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.00226
	0.10	4.18	0.51	0.0847	0.0154	0.0293	0.0251	0.0126	0.00214
0.10	0.20	4.36	0.52	0.0860	0.0158	0.0300	0.0255	0.0126	0.00201
	0.30	4.53	0.54	0.0872	0.0161	0.0301	0.0259	0.0125	0.00188
	0.40	4.70	0.55	0.0882	0.0165	0.0314	0.0262	0.0124	0.00174
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.00226
	0.10	4.35	0.52	0.0857	0.0155	0.0299	0.0254	0.0127	0.00213
0.20	0.20	4.72	0.54	0.0880	0.0161	0.0311	0.0262	0.0126	0.00197
	0.30	5.11	0.56	0.0901	0.0166	0.0324	0.0269	0.0125	0.00178
	0.40	5.51	0.58	0.0921	0.0171	0.0336	0.0276	0.0123	0.00156
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.00226
	0.10	4.49	0.53	0.0863	0.0155	0.0301	0.0257	0.0128	0.00219
0.30	0.20	5.05	0.56	0.0893	0.0160	0.0317	0.0267	0.0128	0.00207
	0.30	5.69	0.59	0.0923	0.0165	0.0334	0.0278	0.0127	0.00190
	0.40	6.41	0.61	0.0951	0.0171	0.0352	0.0287	0.0124	0.00167
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.00226
	0.10	4.61	0.53	0.0866	0.0154	0.0302	0.0259	0.0129	0.00225
0.40	0.20	5.35	0.56	0.0901	0.0158	0.0318	0.0271	0.0131	0.00221
	0.30	6.25	0.60	0.0936	0.0162	0.0337	0.0284	0.0131	0.00211
	0.40	7.37	0.64	0.0971	0.0168	0.0359	0.0297	0.0128	0.00195
					$C_{F1} = 0.5C_{N1}$; $C_{F2} = 0.5C_{N2}$				
0.00	—	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.10	4.16	0.51	0.0857	0.0155	0.0296	0.0254	0.0130	0.0023
0.10	0.20	4.31	0.52	0.0879	0.0158	0.0304	0.0261	0.0133	0.0023
	0.30	4.45	0.54	0.0900	0.0162	0.0312	0.0267	0.0135	0.0023
	0.40	4.58	0.54	0.0918	0.0165	0.0319	0.0273	0.0138	0.0023
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.10	4.30	0.52	0.0872	0.0156	0.0301	0.0259	0.0132	0.0023
0.20	0.20	4.61	0.55	0.0912	0.0161	0.0317	0.0272	0.0138	0.0023
	0.30	4.92	0.57	0.0951	0.0167	0.0332	0.0285	0.0143	0.0024
	0.40	5.23	0.58	0.0989	0.0172	0.0347	0.0298	0.0148	0.0024
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.10	4.43	0.53	0.0881	0.0156	0.0305	0.0263	0.0134	0.0023
0.30	0.20	4.89	0.56	0.0932	0.0161	0.0324	0.0281	0.0142	0.0024
	0.30	5.40	0.59	0.0986	0.0167	0.0345	0.0300	0.0150	0.0024
	0.40	5.93	0.62	0.1042	0.0173	0.0367	0.0320	0.0158	0.0025
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.10	4.54	0.54	0.0884	0.0155	0.0305	0.0265	0.0135	0.0024
0.40	0.20	5.16	0.57	0.0941	0.0159	0.0326	0.0286	0.0145	0.0025
	0.30	5.87	0.61	0.1005	0.0165	0.0350	0.0310	0.0155	0.0025
	0.40	6.67	0.64	0.1076	0.0170	0.0377	0.0336	0.0166	0.0026
					$C_{F1} = 2C_{N1}$; $C_{F2} = 2C_{N2}$				
0.00	—	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
	0.00	4.00	0.50	0.0833	0.0150	0.0287	0.0247	0.0127	0.0023
0.10	0.10	4.27	0.51	0.0817	0.0153	0.0289	0.0241	0.0116	0.0018
	0.20	4.56	0.52	0.0798	0.0156	0.0290	0.0234	0.0103	0.0013
	0.00	4.00	0.50	0.0833	0.0151	0.0287	0.0247	0.0127	0.0023
0.20	0.10	4.49	0.51	0.0819	0.0154	0.0291	0.0240	0.0114	0.0019
	0.20	5.11	0.53	0.0789	0.0158	0.0293	0.0228	0.0096	0.0014

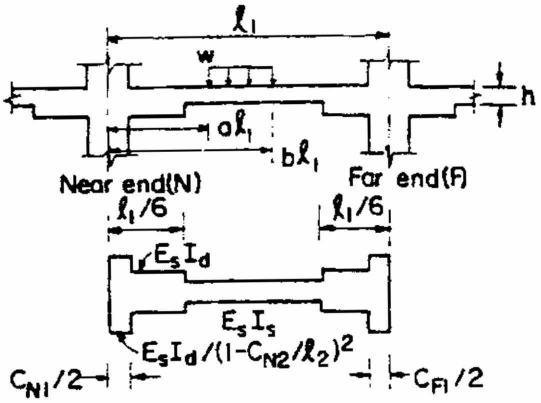
APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS (cont'd)

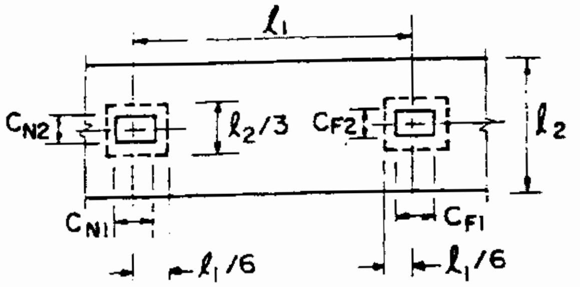
Table A2 Moment Distribution Constants for Slab-Beam Members (Drop thickness = 0.25h)



C_{N1}/ℓ_1	C_{N2}/ℓ_2	Stiffness Factors k_{NF}	Carry Over Factors C_{NF}	Unif. Load Fixed end M. Coeff. (m_{NF})	Fixed end moment Coeff. (m_{NF}) for (b—a) = 0.2				
					a = 0.0	a = 0.2	a = 0.4	a = 0.6	a = 0.8
$C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$									
0.00	—	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
0.10	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	4.99	0.55	0.0890	0.0160	0.0316	0.0266	0.0128	0.0020
	0.20	5.18	0.56	0.0901	0.0163	0.0322	0.0270	0.0127	0.0019
	0.30	5.37	0.57	0.0911	0.0167	0.0328	0.0273	0.0126	0.0018
0.20	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	5.17	0.56	0.0900	0.0161	0.0320	0.0269	0.0128	0.0020
	0.20	5.56	0.58	0.0918	0.0166	0.0332	0.0276	0.0126	0.0018
	0.30	5.96	0.60	0.0936	0.0171	0.0344	0.0282	0.0124	0.0016
0.30	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	5.32	0.57	0.0905	0.0161	0.0323	0.0272	0.0128	0.0021
	0.20	5.90	0.59	0.0930	0.0166	0.0338	0.0281	0.0127	0.0019
	0.30	6.55	0.62	0.0955	0.0171	0.0354	0.0290	0.0124	0.0017
$C_{F1} = 0.5C_{N1}$; $C_{F2} = 0.5C_{N2}$									
0.00	—	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
0.10	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	4.96	0.55	0.0900	0.0160	0.0317	0.0269	0.0131	0.0022
	0.20	5.12	0.56	0.0920	0.0164	0.0325	0.0276	0.0134	0.0022
0.20	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	5.11	0.56	0.0914	0.0162	0.0323	0.0275	0.0133	0.0022
	0.20	5.43	0.58	0.0950	0.0167	0.0337	0.0286	0.0138	0.0022
$C_{F1} = 2C_{N1}$; $C_{F2} = 2C_{N2}$									
0.00	—	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
0.10	0.00	4.79	0.54	0.0879	0.0157	0.0309	0.0263	0.0129	0.0022
	0.10	5.10	0.55	0.0860	0.0159	0.0311	0.0256	0.0117	0.0017

Table A3 Moment Distribution Constants for Slab-Beam Members (Drop thickness = 0.50h)





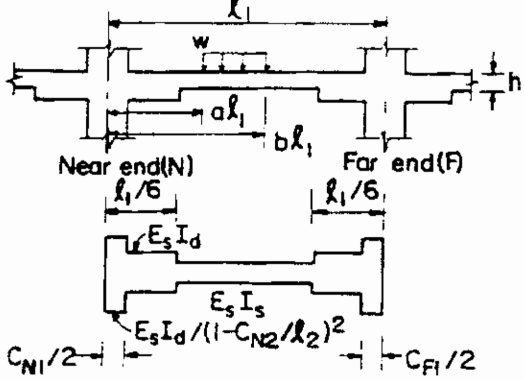
$FEM_{NF} = \sum_{i=1}^{n} m_{NFi} W_i \ell_1^2$

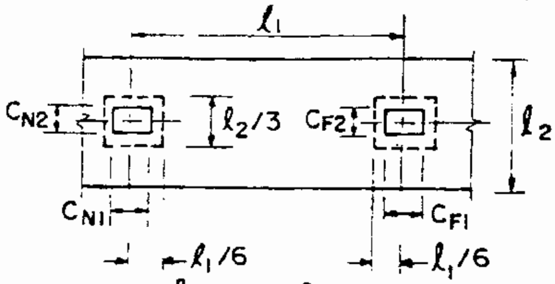
$K_{NF} = k_{NF} E_{cs} I_s / \ell_1$

C_{N1}/ℓ_1	C_{N2}/ℓ_2	Stiffness Factors k_{NF}	Carry Over Factors C_{NF}	Unif. Load Fixed end M. Coeff. (m_{NF})	Fixed end moment Coeff. (m_{NF}) for (b—a) = 0.2				
					a = 0.0	a = 0.2	a = 0.4	a = 0.6	a = 0.8
$C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$									
0.00	—	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
0.10	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.04	0.60	0.0936	0.0167	0.0341	0.0282	0.0126	0.0018
	0.20	6.24	0.61	0.0940	0.0170	0.0347	0.0285	0.0125	0.0017
	0.30	6.43	0.61	0.0952	0.0173	0.0353	0.0287	0.0123	0.0016
0.20	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.22	0.61	0.0942	0.0168	0.0346	0.0285	0.0126	0.0018
	0.20	6.62	0.62	0.0957	0.0172	0.0356	0.0290	0.0123	0.0016
	0.30	7.01	0.64	0.0971	0.0177	0.0366	0.0294	0.0120	0.0014
0.30	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.37	0.61	0.0947	0.0168	0.0348	0.0287	0.0126	0.0018
	0.20	6.95	0.63	0.0967	0.0172	0.0362	0.0294	0.0123	0.0016
	0.30	7.57	0.65	0.0986	0.0177	0.0375	0.0300	0.0119	0.0014
$C_{F1} = 0.5C_{N1}$; $C_{F2} = 0.5C_{N2}$									
0.00	—	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
0.10	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.00	0.60	0.0945	0.0167	0.0343	0.0285	0.0130	0.0020
	0.20	6.16	0.60	0.0962	0.0170	0.0350	0.0291	0.0132	0.0020
0.20	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.15	0.60	0.0957	0.0169	0.0348	0.0290	0.0131	0.0020
	0.20	6.47	0.62	0.0987	0.0173	0.0360	0.0300	0.0134	0.0020
$C_{F1} = 2C_{N1}$; $C_{F2} = 2C_{N2}$									
0.00	—	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
0.10	0.00	5.84	0.59	0.0926	0.0164	0.0335	0.0279	0.0128	0.0020
	0.10	6.17	0.60	0.0907	0.0166	0.0337	0.0273	0.0116	0.0015

APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS (cont'd)

Table A4 Moment Distribution Constants for Slab-Beam Members (Drop thickness = 0.75h)





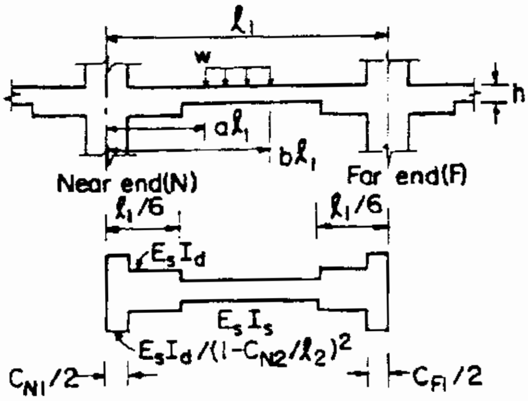
$FEM_{NF} = \sum_{i=1}^{n} m_{NFi} W_i \ell_1^2$

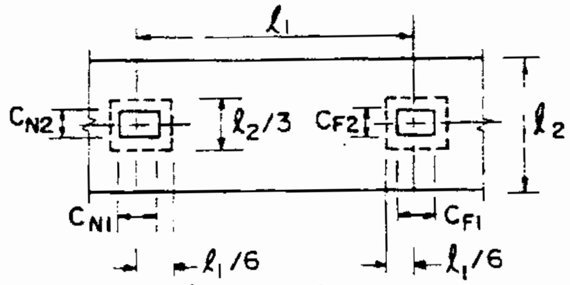
$K_{NF} = k_{NF} E_{cs} I_s / \ell_1$

C_{N1}/ℓ_1	C_{N2}/ℓ_2	Stiffness Factors k_{NF}	Carry Over Factors C_{NF}	Unif. Load Fixed end M. Coeff. (m_{NF})	Fixed end moment Coeff. (m_{NF}) for (b—a) = 0.2				
					a = 0.0	a = 0.2	a = 0.4	a = 0.6	a = 0.8
$C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$									
0.00	—	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
0.10	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.12	0.64	0.0972	0.0174	0.0365	0.0295	0.0122	0.0016
	0.20	7.31	0.64	0.0978	0.0176	0.0370	0.0297	0.0120	0.0014
	0.30	7.48	0.65	0.0984	0.0179	0.0375	0.0299	0.0118	0.0013
0.20	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.12	0.64	0.0977	0.0175	0.0369	0.0297	0.0121	0.0015
	0.20	7.31	0.65	0.0988	0.0178	0.0378	0.0301	0.0118	0.0013
	0.30	7.48	0.67	0.0999	0.0182	0.0386	0.0304	0.0115	0.0011
0.30	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.29	0.65	0.0981	0.0175	0.0371	0.0299	0.0121	0.0015
	0.20	7.66	0.66	0.0996	0.0179	0.0383	0.0304	0.0117	0.0013
	0.30	8.02	0.68	0.1009	0.0182	0.0394	0.0309	0.0113	0.0011
$C_{F1} = 0.5C_{N1}$; $C_{F2} = 0.5C_{N2}$									
0.00	—	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
0.10	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.08	0.64	0.0980	0.0174	0.0366	0.0298	0.0125	0.0017
	0.20	7.23	0.64	0.0993	0.0177	0.0372	0.0302	0.0126	0.0016
0.20	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.21	0.64	0.0991	0.0175	0.0371	0.0302	0.0126	0.0017
	0.20	7.51	0.65	0.1014	0.0179	0.0381	0.0310	0.0128	0.0016
$C_{F1} = 2C_{N1}$; $C_{F2} = 2C_{N2}$									
0.00	—	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
0.10	0.00	6.92	0.63	0.0965	0.0171	0.0360	0.0293	0.0124	0.0017
	0.10	7.26	0.64	0.0946	0.0173	0.0361	0.0287	0.0112	0.0013

APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS (cont'd)

Table A5 Moment Distribution Constants for Slab-Beam Members (Drop thickness = h)





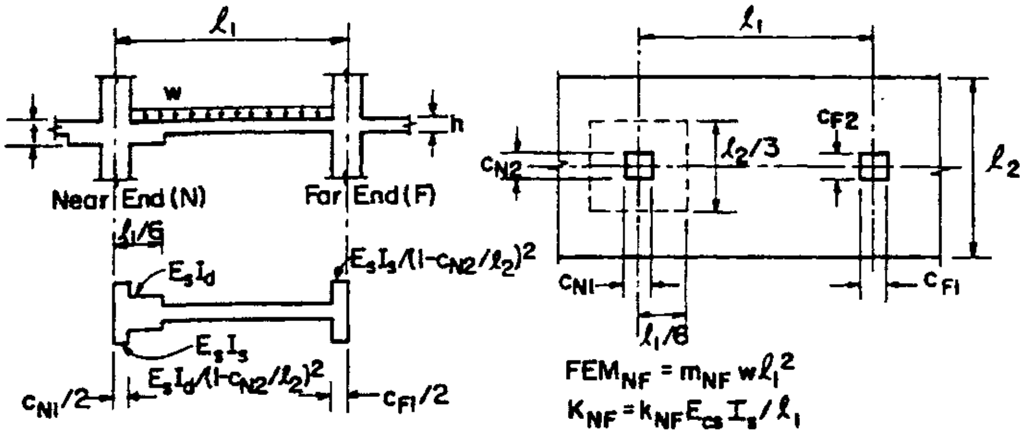
$$FEM_{NF} = \sum_{i=1}^{n} m_{NFi} W_i \ell_1^2$$

$$K_{NF} = k_{NF} E_{cs} I_s / \ell_1$$

C_{N1}/ℓ_1	C_{N2}/ℓ_2	Stiffness Factors k_{NF}	Carry Over Factors C_{NF}	Unif. Load Fixed end M. Coeff. (m_{NF})	Fixed end moment Coeff. (m_{NF}) for (b—a) = 0.2				
					a = 0.0	a = 0.2	a = 0.4	a = 0.6	a = 0.8
$C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$									
0.00	—	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
0.10	0.00	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.07	0.66	0.0998	0.0180	0.0385	0.0305	0.0116	0.0013
	0.20	8.24	0.67	0.1003	0.0182	0.0389	0.0306	0.0115	0.0012
	0.30	8.40	0.67	0.1007	0.0183	0.0393	0.0307	0.0113	0.0011
0.20	0.00	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.22	0.67	0.1002	0.0180	0.0388	0.0306	0.0115	0.0012
	0.20	8.55	0.68	0.1010	0.0183	0.0395	0.0309	0.0112	0.0011
	0.30	9.87	0.69	0.1018	0.0186	0.0402	0.0311	0.0109	0.0009
0.30	0.00	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.35	0.67	0.1005	0.0181	0.0390	0.0307	0.0115	0.0012
	0.20	8.82	0.68	0.1016	0.0184	0.0399	0.0311	0.0111	0.0011
	0.30	9.28	0.70	0.1026	0.0187	0.0409	0.0314	0.0107	0.0009
$C_{F1} = 0.5C_{N1}$; $C_{F2} = 0.5C_{N2}$									
0.00	—	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
0.10	0.00	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.03	0.66	0.1006	0.0180	0.0386	0.0307	0.0119	0.0014
	0.20	8.16	0.67	0.1016	0.0182	0.0390	0.0310	0.0120	0.0014
0.20	0.00	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.15	0.67	0.1014	0.0181	0.0389	0.0310	0.0120	0.0014
	0.20	8.41	0.68	0.1032	0.0184	0.0398	0.0316	0.0121	0.0013
$C_{F1} = 2C_{N1}$; $C_{F2} = 0.5C_{N2}$									
0.00	—	7.89	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
0.10	0.00	7.79	0.66	0.0993	0.0177	0.0380	0.0303	0.0118	0.0014
	0.10	8.20	0.67	0.0981	0.0179	0.0382	0.0297	0.0113	0.0010

APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS (cont'd)

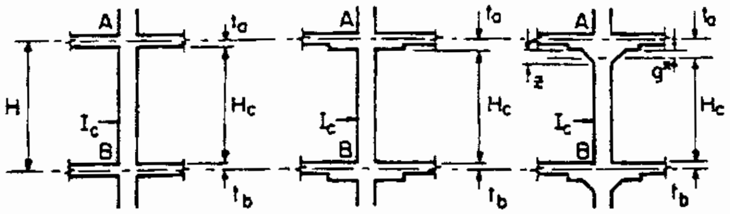
Table A6 Moment Distribution Constants for Slab-Beam Members
(Column dimensions assumed equal at near end and far end —$c_{F1} = c_{N1}$, $c_{F2} = c_{N2}$)



C_1/ℓ_1	C_2/ℓ_2	t = 1.5h						t = 2h					
		k_{NF}	C_{NF}	m_{NF}	k_{FN}	C_{FN}	m_{FN}	k_{NF}	C_{NF}	m_{NF}	k_{FN}	C_{FN}	m_{FN}
0.00	—	5.39	0.49	0.1023	4.26	0.60	0.0749	6.63	0.49	0.1190	4.49	0.65	0.0676
0.10	0.00	5.39	0.49	0.1023	4.26	0.60	0.0749	6.63	0.49	0.1190	4.49	0.65	0.0676
	0.10	5.65	0.52	0.1012	4.65	0.60	0.0794	7.03	0.54	0.1145	5.19	0.66	0.0757
	0.20	5.86	0.54	0.1012	4.91	0.61	0.0818	7.22	0.56	0.1140	5.43	0.67	0.0778
	0.30	6.05	0.55	0.1025	5.10	0.62	0.0838	7.36	0.56	0.1142	5.57	0.67	0.0786
0.20	0.00	5.39	0.49	0.1023	4.26	0.60	0.0749	6.63	0.49	0.1190	4.49	0.65	0.0676
	0.10	5.88	0.54	0.1006	5.04	0.61	0.0826	7.41	0.58	0.1111	5.96	0.66	0.0823
	0.20	6.33	0.58	0.1003	5.63	0.62	0.0874	7.85	0.61	0.1094	6.57	0.67	0.0872
	0.30	6.75	0.60	0.1008	6.10	0.64	0.0903	8.18	0.63	0.1093	6.94	0.68	0.0892
0.30	0.00	5.39	0.49	0.1023	4.26	0.60	0.075	6.63	0.49	0.1190	4.49	0.65	0.0676
	0.10	6.08	0.56	0.1003	5.40	0.61	0.085	7.76	0.62	0.1087	6.77	0.67	0.0873
	0.20	6.78	0.61	0.0996	6.38	0.63	0.092	8.49	0.66	0.1055	7.91	0.68	0.0952
	0.30	7.48	0.64	0.0997	7.25	0.65	0.096	9.06	0.68	0.1047	8.66	0.69	0.0991

APPENDIX 20A DESIGN AIDS FOR MOMENT DISTRIBUTION CONSTANTS (cont'd)

Table A7 Stiffness and Carry-Over Factors for Columns



$K_c = k \frac{EI_c}{H}$

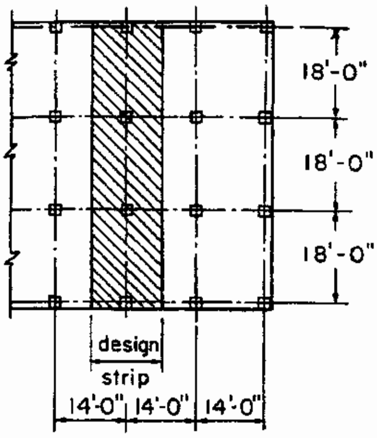
For values of k_{BA} and c_{BA} read (t_a/t_b) as (t_b/t_a)

*g can be approximated as z/2.

t_a/t_b \ H/H_c		1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50
0.00	kAB	4.20	4.40	4.60	4.80	5.00	5.20	5.40	5.60	5.80	6.00
	CAB	0.57	0.65	0.73	0.80	0.87	0.95	1.03	1.10	1.17	1.25
0.2	kAB	4.31	4.62	4.95	5.30	5.65	6.02	6.40	6.79	7.20	7.62
	CAB	0.56	0.62	0.68	0.74	0.80	0.85	0.91	0.96	1.01	1.07
0.4	kAB	4.38	4.79	5.22	5.67	6.15	6.65	7.18	7.74	8.32	8.94
	CAB	0.55	0.60	0.65	0.70	0.74	0.79	0.83	0.87	0.91	0.94
0.6	kAB	4.44	4.91	5.42	5.96	6.54	7.15	7.81	8.50	9.23	10.01
	CAB	0.55	0.59	0.63	0.67	0.70	0.74	0.77	0.80	0.83	0.85
0.8	kAB	4.49	5.01	5.58	6.19	6.85	7.56	8.31	9.12	9.98	10.89
	CAB	0.54	0.58	0.61	0.64	0.67	0.70	0.72	0.75	0.77	0.79
1.0	kAB	4.52	5.09	5.71	6.38	7.11	7.89	8.73	9.63	10.60	11.62
	CAB	0.54	0.57	0.60	0.62	0.65	0.67	0.69	0.71	0.73	0.74
1.2	kAB	4.55	5.16	5.82	6.54	7.32	8.17	9.08	10.07	11.12	12.25
	CAB	0.53	0.56	0.59	0.61	0.63	0.65	0.66	0.68	0.69	0.70
1.4	kAB	4.58	5.21	5.91	6.68	7.51	8.41	9.38	10.43	11.57	12.78
	CAB	0.53	0.55	0.58	0.60	0.61	0.63	0.64	0.65	0.66	0.67
1.6	kAB	4.60	5.26	5.99	6.79	7.66	8.61	9.64	10.75	11.95	13.24
	CAB	0.53	0.55	0.57	0.59	0.60	0.61	0.62	0.63	0.64	0.65
1.8	kAB	4.62	5.30	6.06	6.89	7.80	8.79	9.87	11.03	12.29	13.65
	CAB	0.52	0.55	0.56	0.58	0.59	0.60	0.61	0.61	0.62	0.63
2.0	kAB	4.63	5.34	6.12	6.98	7.92	8.94	10.06	11.27	12.59	14.00
	CAB	0.52	0.54	0.56	0.57	0.58	0.59	0.59	0.60	0.60	0.61
2.2	kAB	4.65	5.37	6.17	7.05	8.02	9.08	10.24	11.49	12.85	14.31
	CAB	0.52	0.54	0.55	0.56	0.57	0.58	0.58	0.59	0.59	0.59
2.4	kAB	4.66	5.40	6.22	7.12	8.11	9.20	10.39	11.68	13.08	14.60
	CAB	0.52	0.53	0.55	0.56	0.56	0.57	0.57	0.58	0.58	0.58
2.6	kAB	4.67	5.42	6.26	7.18	8.20	9.31	10.53	11.86	13.29	14.85
	CAB	0.52	0.53	0.54	0.55	0.56	0.56	0.56	0.57	0.57	0.57
2.8	kAB	4.68	5.44	6.29	7.23	8.27	9.41	10.66	12.01	13.48	15.07
	CAB	0.52	0.53	0.54	0.55	0.55	0.55	0.56	0.56	0.56	0.56
3.0	kAB	4.69	5.46	6.33	7.28	8.34	9.50	10.77	12.15	13.65	15.28
	CAB	0.52	0.53	0.54	0.54	0.55	0.55	0.55	0.55	0.55	0.55
3.2	kAB	4.70	5.48	6.36	7.33	8.40	9.58	10.87	12.28	13.81	15.47
	CAB	0.52	0.53	0.53	0.54	0.54	0.54	0.54	0.54	0.54	0.54
3.4	kAB	4.71	5.50	6.38	7.37	8.46	9.65	10.97	12.40	13.95	15.64
	CAB	0.51	0.52	0.53	0.53	0.54	0.54	0.54	0.53	0.53	0.53
3.6	kAB	4.71	5.51	6.41	7.41	8.51	9.72	11.05	12.51	14.09	15.80
	CAB	0.51	0.52	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.52
3.8	kAB	4.72	5.53	6.43	7.44	8.56	9.78	11.13	12.60	14.21	15.95
	CAB	0.51	0.52	0.53	0.53	0.53	0.53	0.53	0.52	0.52	0.52
4.0	kAB	4.72	5.54	6.45	7.47	8.60	9.84	11.21	12.70	14.32	16.08
	CAB	0.51	0.52	0.52	0.53	0.53	0.52	0.52	0.52	0.52	0.51
4.2	kAB	4.73	5.55	6.47	7.50	8.64	9.90	11.27	12.78	14.42	16.20
	CAB	0.51	0.52	0.52	0.52	0.52	0.52	0.52	0.51	0.51	0.51
4.4	kAB	4.73	5.56	6.49	7.53	8.68	9.95	11.34	12.86	14.52	16.32
	CAB	0.51	0.52	0.52	0.52	0.52	0.52	0.51	0.51	0.51	0.50
4.6	kAB	4.74	5.57	6.51	7.55	8.71	9.99	11.40	12.93	14.61	16.43
	CAB	0.51	0.52	0.52	0.52	0.52	0.52	0.51	0.51	0.50	0.50
4.8	kAB	4.74	5.58	6.53	7.58	8.75	10.03	11.45	13.00	14.69	16.53
	CAB	0.51	0.52	0.52	0.52	0.52	0.51	0.51	0.50	0.50	0.49
5.0	kAB	4.75	5.59	6.54	7.60	8.78	10.07	11.50	13.07	14.77	16.62
	CAB	0.51	0.51	0.52	0.52	0.51	0.51	0.51	0.50	0.49	0.49
6.0	kAB	4.76	5.63	6.60	7.69	8.90	10.24	11.72	13.33	15.10	17.02
	CAB	0.51	0.51	0.51	0.51	0.50	0.50	0.49	0.49	0.48	0.47
7.0	kAB	4.78	5.66	6.65	7.76	9.00	10.37	11.88	13.54	15.35	17.32
	CAB	0.51	0.51	0.51	0.50	0.50	0.49	0.48	0.48	0.47	0.46
8.0	kAB	4.78	5.68	6.69	7.82	9.07	10.47	12.01	13.70	15.54	17.56
	CAB	0.51	0.51	0.50	0.50	0.49	0.49	0.48	0.47	0.46	0.45
9.0	kAB	4.79	5.69	6.71	7.86	9.13	10.55	12.11	13.83	15.70	17.74
	CAB	0.50	0.50	0.50	0.50	0.49	0.48	0.47	0.46	0.45	0.45
10.0	kAB	4.80	5.71	6.74	7.89	9.18	10.61	12.19	13.93	15.83	17.90
	CAB	0.50	0.50	0.50	0.49	0.48	0.48	0.47	0.46	0.45	0.44

Example 20.1—Two-Way Slab Without Beams Analyzed by Equivalent Frame Method

Using the Equivalent Frame Method, determine design moments for the slab system in the direction shown, for an intermediate floor.



Story height = 9 ft
Column dimensions = 16 × 16 in.
Lateral loads to be resisted by shear walls
No edge beams
Partition weight = 20 psf
Service live load = 40 psf
f'_c = 4,000 psi (for slabs), normal weight concrete
f'_c = 6,000 psi (for columns), normal weight concrete
f_y = 60,000 psi

Calculations and Discussion	Code Reference

1. Preliminary design for slab thickness h:

   a. Control of deflections.

   For flat plate slab systems, the minimum overall thickness h with Grade 60 reinforcement is (see Table 18-1): *9.5.3.2*

   $$h = \frac{\ell_n}{30} = \frac{200}{30} = 6.67 \text{ in.}$$ *Table 9.5 (a)*

   but not less than 5 in. *9.5.3.2(a)*

   where ℓ_n = length of clear span in the long direction = 216 - 16 = 200 in.

   Try 7 in. slab for all panels (weight = 87.5 psf)

   Note, in addition to ACI 318-05 deflection control requirements, thickness of slab should satisfy the minimum required for fire resistance, as specified in the locally adopted building code.

   b. Shear strength of slab.

   Use average effective depth d = 5.75 in. (3/4 in. cover and No. 4 bar)

   Factored dead load, q_{Du} = 1.2 (87.5 + 20) = 129 psf *9.2.1*
   Factored live load, q_{Lu} = 1.6 × 40 = 64 psf
   Total factored load = 193 psf

   For wide beam action consider a 12-in. wide strip taken at d distance from the face of support in the long direction (see Fig. 20-13). *11.12.1.1*

   $$V_u = 0.193 \times 7.854 = 1.5 \text{ kips}$$

   $$V_c = 2\sqrt{f'_c}\, b_w d$$

$\phi V_c = 0.75 \times 2\sqrt{4,000} \times 12 \times 5.75/1,000 = 6.6 \text{ kips} > V_u$ O.K. *9.3.2.3*

For two-way action, since there are no shear forces at the centerlines of adjacent panels, the shear strength at d/2 distance around the support is computed as follows:

$V_u = 0.193 [(18 \times 14) - 1.81^2] = 48.0 \text{ kips}$

$V_c = 4\sqrt{f'_c}\, b_o d$ (for square interior column) *Eq. (11-35)*

$= 4\sqrt{4,000}\ (4 \times 21.75) \times 5.75/1,000 = 126.6 \text{ kips}$

$\phi V_c = 0.75 \times 126.6 = 95.0 \text{ kips} > V_u$ O.K. *9.3.2.3*

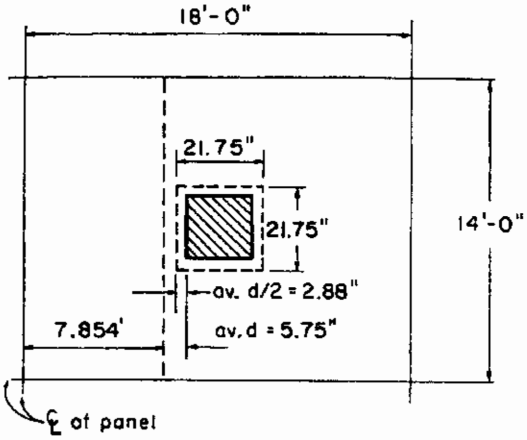


Figure 20-13 Critical Sections for Shear for Example Problem

Preliminary design indicates that a 7 in. overall slab thickness is adequate for control of deflections and shear strength.

2. Frame members of equivalent frame:

Determine moment distribution factors and fixed-end moments for the equivalent frame members. The moment distribution procedure will be used to analyze the partial frame. Stiffness factors k, carry over factors COF, and fixed-end moment factors FEM for the slab-beams and column members are determined using the tables of Appendix 20-A. These calculations are shown here.

a. Flexural stiffness of slab-beams at both ends, K_{sb}.

$$\frac{c_{N1}}{\ell_1} = \frac{16}{(18 \times 12)} = 0.07, \quad \frac{c_{N2}}{\ell_2} = \frac{16}{(14 \times 12)} = 0.1$$

For $c_{F1} = c_{N1}$ and $c_{F2} = c_{N2}$, $k_{NF} = k_{FN} = 4.13$ by interpolation from Table A1 in Appendix 20A.

Thus, $K_{sb} = k_{NF} \dfrac{E_{cs}I_s}{\ell_1} = 4.13 \dfrac{E_{cs}I_s}{\ell_1}$ *Table A1*

$= 4.13 \times 3.60 \times 10^6 \times 4{,}802/216 = 331 \times 10^6$ in.-lb

where $I_s = \dfrac{\ell_2 h^3}{12} = \dfrac{168\,(7)^3}{12} = 4{,}802$ in.^{4}

$E_{cs} = 57{,}000\sqrt{f'_c} = 57{,}000\sqrt{4{,}000} = 3.60 \times 10^6$ psi *8.5.1*

Carry-over factor COF = 0.509, by interpolation from Table A1.

Fixed-end moment FEM = $0.0843 w_u \ell_2 \ell_1^2$, by interpolation from Table A1.

b. Flexural stiffness of column members at both ends, K_c.

Referring to Table A7, Appendix 20A, $t_a = 3.5$ in., $t_b = 3.5$ in.,

$H = 9$ ft $= 108$ in., $H_c = 101$ in., $t_a/t_b = 1$, $H/H_c = 1.07$

Thus, $k_{AB} = k_{BA} = 4.74$ by interpolation.

$K_c = 4.74 E_{cc} I_c / \ell_c$ *Table A7*

$= 4.74 \times 4.42 \times 10^6 \times 5461/108 = 1059 \times 10^6$ in.-lb

where $I_c = \dfrac{c^4}{12} = \dfrac{(16)^4}{12} = 5{,}461$ in.^{4}

$E_{cs} = 57{,}000\sqrt{f'_c} = 57{,}000\sqrt{6{,}000} = 4.42 \times 10^6$ psi *8.5.1*

$\ell_c = 9$ ft $= 108$ in.

c. Torsional stiffness of torsional members, K_t.

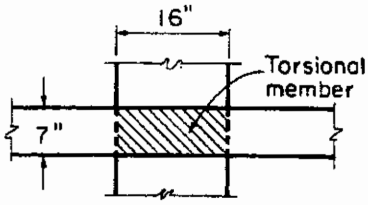
$$K_t = \frac{9E_{cs}C}{\left[\ell_2 (1 - c_2/\ell_2)^3\right]}$$ *R.13.7.5*

$$= \frac{9 \times 3.60 \times 10^6 \times 13.25}{168(0.905)^3} = 3.45 \times 10^6 \text{ in.-lb}$$

where $C = \Sigma\ (1 - 0.63\ x/y)\ (x^3y/3)$ *13.0*

$= (1 - 0.63 \times 7/16)\ (7^3 \times 16/3) = 1{,}325\ \text{in.}^4$

$c_2 = 16$ in. and $\ell_2 = 14$ ft $= 168$ in.



Condition (a) of Fig. 20-7

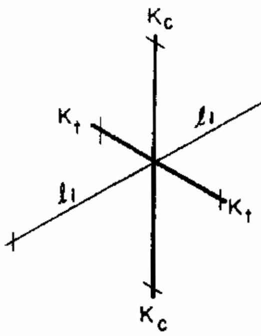
d. Equivalent column stiffness K_{ec}.

$$K_{ec} = \frac{\Sigma K_c \times \Sigma K_t}{\Sigma K_c + \Sigma K_t}$$

$$= \frac{(2 \times 1{,}059)(2 \times 345)}{[(2 \times 1{,}059) + (2 \times 345)]}$$

$$= 520 \times 10^6\ \text{in.-lb}$$

where ΣK_t is for two torsional members, one on each side of column, and ΣK_c is for the upper and lower columns at the slab-beam joint of an intermediate floor.



e. Slab-beam joint distribution factors DF.

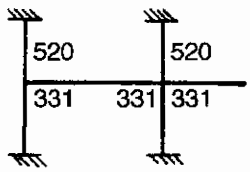
At exterior joint,

$$DF = \frac{331}{(331 + 520)} = 0.389$$

At interior joint,

$$DF = \frac{331}{(331 + 331 + 520)} = 0.280$$

COF for slab-beam = 0.509



3. Partial frame analysis of equivalent frame:

Determine maximum negative and positive moments for the slab-beams using the moment distribution method. Since the service live load does not exceed three-quarters of the service dead load, design moments are assumed to occur at all critical sections with full factored live load on all spans. *13.7.6.2*

$$\frac{L}{D} = \frac{40}{(87.5 + 20)} = 0.37 < \frac{3}{4}$$

a. Factored load and fixed-end moments.

Factored dead load $q_{Du} = 1.2\ (87.5 + 20) = 129$ psf *Eq. (9-2)*

Factored live load $q_{Lu} = 1.6\ (40) = 64$ psf *Eq. (9-2)*

Factored load $q_u = q_{Du} + q_{Lu} = 193$ psf

FEM's for slab-beams $= m_{NF} q_u \ell_2 \ell_1^2$ (Table A1, Appendix 20A)
$= 0.0843\,(0.193 \times 14)\,18^2 = 73.8$ ft-kips

b. Moment distribution. Computations are shown in Table 20-1. Counterclockwise rotational moments acting on the member ends are taken as positive. Positive span moments are determined from the following equation:

M_u (midspan) $= M_o - (M_{uL} + M_{uR})/2$

where M_o is the moment at midspan for a simple beam.

When the end moments are not equal, the maximum moment in the span does not occur at midspan, but its value is close to that at midspan for this example.

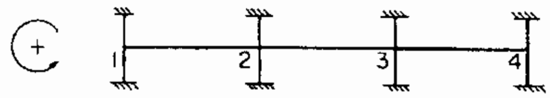
Positive moment in span 1-2:

$+M_u = (0.193 \times 14)\,18^2/8 - (46.6 + 84.0)/2 = 44.1$ ft-kips

Positive moment in span 2-3:

$+M_u = (0.193 \times 14)\,18^2/8 - (76.2 + 76.2)/2 = 33.2$ ft-kips

Table 20-1 Moment Distribution for Partial Frame



Joint	1	2		3		4
Member	1-2	2-1	2-3	3-2	3-4	4-3
DF	0.389	0.280	0.280	0.280	0.280	0.389
COF	0.509	0.509	0.509	0.509	0.509	0.509
FEM	+73.8	-73.8	+73.8	-73.8	+73.8	-73.8
Dist	-28.7	0.0	0.0	0.0	0.0	28.7
CO	0.0	-14.6	0.0	0.0	14.6	0.0
Dist	0.0	4.1	4.1	-4.1	-4.1	0.0
CO	2.1	0.0	-2.1	2.1	0.0	-2.1
Dist	-0.8	0.6	0.6	-0.6	-0.6	0.8
CO	0.3	-0.4	-0.3	0.3	0.4	-0.3
Dist	-0.1	0.2	0.2	-0.2	-0.2	0.1
CO	0.1	-0.1	-0.1	0.1	0.1	-0.1
Dist	0.0	0.0	0.0	0.0	0.0	0.0
Neg. M	46.6	-84.0	76.2	-76.2	84.0	-46.6
M @ midspan	44.1		33.2		44.1	

4. Design moments:

Positive and negative factored moments for the slab system in the direction of analysis are plotted in Fig. 20-14. The negative design moments are taken at the faces of rectilinear supports but not at distances greater than $0.175\ell_1$ from the centers of supports. *13.7.7.1*

$\frac{16 \text{ in.}}{2} = 0.67 \text{ ft} < 0.175 \times 18 = 3.2 \text{ ft}$ (Use face of support location)

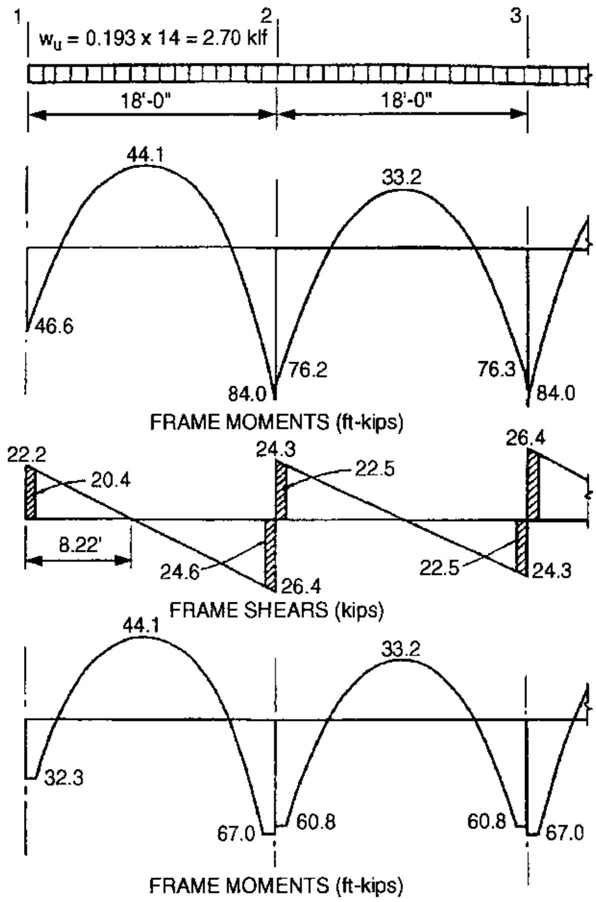


Figure 20-14 Positive and Negative Design Moments for Slab-Beam (All Spans Loaded with Full Factored Live Load)

5. Total factored moment per span:

Slab systems within the limitations of 13.6.1 may have the resulting moments reduced in such proportion that the numerical sum of the positive and average negative moments need not be greater than: *13.7.7.4*

$$M_o = \frac{q_u \ell_2 \ell_n^2}{8} = 0.193 \times 14 \times (16.67)^2/8 = 93.9 \text{ ft-kips}$$

End spans: 44.1 + (32.3 + 67.0)/2 = 93.8 ft-kips

Interior span: 33.2 + (60.8 + 60.8)/2 = 94 ft-kips

It may be seen that the total design moments from the Equivalent Frame Method yield a static moment equal to that given by the static moment expression used with the Direct Design Method.

6. Distribution of design moments across slab-beam strip: *13.7.7.5*

The negative and positive factored moments at critical sections may be distributed to the column strip and the two half-middle strips of the slab-beam according to the proportions specified in 13.6.4 and 13.6.6. The requirement of 13.6.1.6 does not apply for slab systems without beams, $\alpha = 0$. Distribution of factored moments at critical sections is summarized in Table 20-2.

Table 20-2 Distribution of Factored Moments

	Factored Moment (ft-kips)	Column Strip		Moment (ft-kips) in Two Half-Middle Strips**
		Percent*	Moment (ft-kips)	
End Span:				
Exterior Negative	32.3	100	32.3	0.0
Positive	44.1	60	26.5	17.7
Interior Negative	67.0	75	50.3	16.7
Interior Span:				
Negative	60.8	75	45.6	15.2
Positive	33.2	60	19.9	13.2

** For slab systems without beams*
*** That portion of the factored moment not resisted by the column strip is assigned to the two half-middle strips.*

7. Column moments:

The unbalanced moment from the slab-beams at the supports of the equivalent frame are distributed to the actual columns above and below the slab-beam in proportion to the relative stiffnesses of the actual columns. Referring to Fig. 20-14, the unbalanced moment at joints 1 and 2 are:

Joint 1 = +46.6 ft-kips

Joint 2 = -84.0 + 76.2 = -7.8 ft-kips

The stiffness and carry-over factors of the actual columns and the distribution of the unbalanced moments to the exterior and interior columns are shown in Fig. 20-15. The design moments for the columns may be taken at the juncture of column and slab.

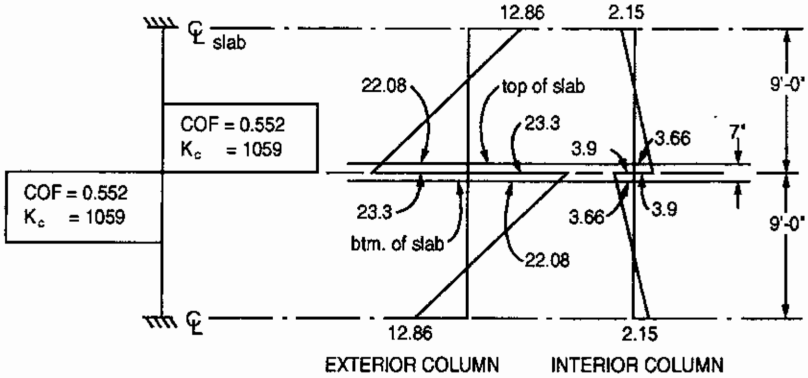


Figure 20-15 Column Moments (Unbalanced Moments from Slab-Beam)

In summary:

Design moment in exterior column = 22.08 ft-kips

Design moment in interior column = 3.66 ft-kips

8. Check slab flexural and shear strength at exterior column

   a. Total flexural reinforcement required for design strip:

      i. Determine reinforcement required for column strip moment $M_u = 32.3$ ft-kips

      Assume tension-controlled section ($\phi = 0.9$) *9.3.2.1*

      Column strip width $b = \dfrac{14 \times 12}{2} = 84$ in. *13.2.1*

      $$R_u = \frac{M_u}{\phi b d^2} = \frac{32.3 \times 12{,}000}{0.9 \times 84 \times 5.75^2} = 155 \text{ psi}$$

      $$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_u}{0.85 f'_c}}\right)$$

      $$= \frac{0.85 \times 4}{60}\left(1 - \sqrt{1 - \frac{2 \times 155}{0.85 \times 4{,}000}}\right) = 0.0026$$

      $A_s = \rho bd = 0.0026 \times 84 \times 5.75 = 1.28$ in.^{2}

      $\rho_{min} = 0.0018$ *13.3.1*

      Min $A_s = 0.0018 \times 84 \times 7 = 1.06$ in.$^2 < 1.28$ in.^{2}

      Number of No. 4 bars $= \dfrac{1.28}{0.2} = 6.4$, say 7 bars

      Maximum spacing $s_{max} = 2h = 14$ in. < 18 in. *13.3.2*

$$c = \frac{a}{\beta_1} = \frac{0.29}{0.85} = 0.34 \text{ in.}$$

$$\epsilon_t = \left(\frac{0.003}{c}\right)d_t - 0.003$$

$$= \left(\frac{0.003}{0.34}\right)5.75 - 0.003 = 0.048 > 0.005$$

Therefore, section is tension-controlled. *10.3.4*

Use 7-No. 4 bars in column strip.

ii. Check slab reinforcement at exterior column for moment transfer between slab and column

Portion of unbalanced moment transferred by flexure = $\gamma_f M_u$ *13.5.3.2*

From Fig. 16-13, Case C:

$$b_1 = c_1 + \frac{d}{2} = 16 + \frac{5.75}{2} = 18.88 \text{ in.}$$

$$b_2 = c_2 + d = 16 + 5.75 = 21.75 \text{ in.}$$

$$\gamma_f = \frac{1}{1 + (2/3)\sqrt{b_1 / b_2}}$$ *Eq. (13-1)*

$$= \frac{1}{1 + (2/3)\sqrt{18.88/21.75}} = 0.62$$

$\gamma_f M_u = 0.62 \times 32.3 = 20.0$ ft-kips

Note that the provisions of 13.5.3.3 may be utilized; however, they are not in this example.

Assuming tension-controlled behavior, determine required area of reinforcement for $\gamma_f M_u = 20.0$ ft-kips

Effective slab width $b = c_2 + 3h = 16 + 3(7) = 37$ in. *13.5.3.2*

$$R_u = \frac{M_u}{\phi b d^2} = \frac{20 \times 12{,}000}{0.9 \times 37 \times 5.75^2} = 218 \text{ psi}$$

$$\rho = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_u}{0.85f'_c}}\right)$$

$$= \frac{0.85 \times 4}{60}\left(1-\sqrt{1-\frac{2 \times 218}{0.85 \times 4{,}000}}\right) = 0.0038$$

$A_s = 0.0038 \times 37 \times 5.75 = 0.80 \text{ in.}^2$

Min. $A_s = 0.0018 \times 37 \times 7 = 0.47 \text{ in.}^2 < 0.80 \text{ in.}^2$ *13.3.1*

Number of No. 4 bars = $\dfrac{0.80}{0.20} = 4$

Verify tension-controlled section:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(4 \times 0.2) \times 60}{0.85 \times 4 \times 37} = 0.38 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{0.38}{0.85} = 0.45 \text{ in.}$$

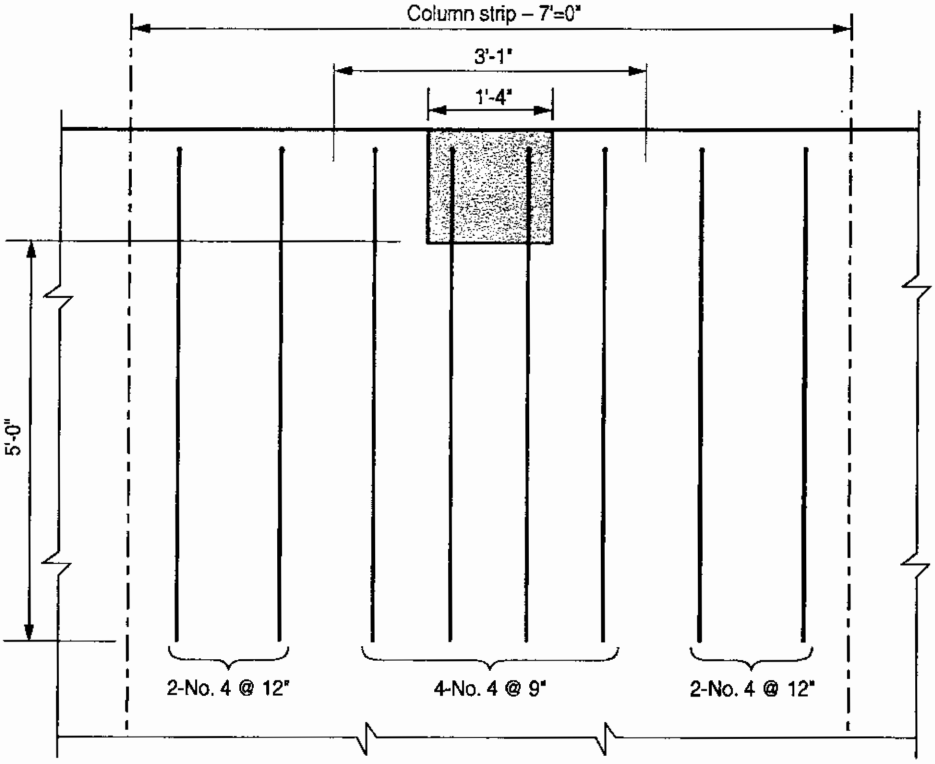
$$\epsilon_t = \left(\frac{0.003}{0.45}\right)5.75 - 0.003 = 0.035 > 0.005$$

Therefore, section is tension-controlled. *10.3.4*

Provide the required 4-No. 4 bars by concentrating 4 of the column strip bars (7-No. 4) within the 37 in. slab width over the column. For symmetry, add one additional No. 4 bar outside of 37-in. width.

Note that the column strip section remains tension-controlled with the addition of 1-No. 4 bar.

The reinforcement details at the edge column are shown below.



iii. Determine reinforcement required for middle strip.

Provide minimum reinforcement, since $M_u = 0$ (see Table 20-2).

Min. $A_s = 0.0018 \times 84 \times 7 = 1.06$ in.^{2}

Maximum spacing $s_{max} = 2h = 14$ in. < 18 in. *13.3.2*

Provide No. 4 @ 14 in. in middle strip.

b. Check combined shear stress at inside face of critical transfer section *11.12.6.1*

For shear strength equations, see Part 16.

$$v_u = \frac{V_u}{A_c} + \frac{\gamma_v M_u}{J/C}$$

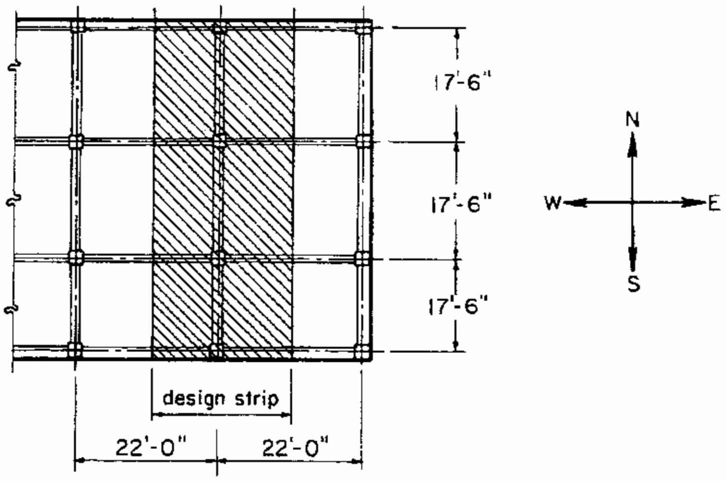
From Example 19.1, $V_u = 25.6$ kips

When factored moments are determined by an accurate method of frame analysis, such as the Equivalent Frame Method, unbalanced moment is taken directly from the results of the frame analysis. Also, considering the approximate nature of the moment transfer analysis procedure, assume the unbalanced moment M_u is at the centroid of the critical transfer section.

Thus, M_u = 32.3 ft-kips (see Table 20-2)	
$\gamma_v = 1 - \gamma_f = 1\text{-}062 = 0.38$	*Eq. (11-39)*
From Example 19.1, critical section properties:	
$A_c = 342.2 \text{ in.}^2$	
$J/c = 2{,}357 \text{ in.}^3$	
$v_u = \dfrac{25{,}600}{342.2} + \dfrac{0.38 \times 32.3 \times 12{,}000}{2{,}357}$	
$= 74.8 + 62.5 = 137.3$ psi	
Allowable shear stress $\phi v_n = \phi 4\sqrt{f'_c} = 189.7 \text{ psi} > v_u$ O.K.	*11.12.6.2*

Example 20.2—Two-Way Slab with Beams Analyzed by Equivalent Frame Method

Using the Equivalent Frame Method, determine design moments for the slab system in the direction shown, for an intermediate floor.



Story height = 12 ft
Edge beam dimensions = 14 × 27 in.
Interior beam dimensions = 14 × 20 in.
Column dimensions = 18 × 18 in.
Service live load = 100 psf

f'_c = 4,000 psi (for all members), normal weight concrete
f_y = 60,000 psi

Calculations and Discussion	Code Reference

1. Preliminary design for slab thickness h.

Control of deflections: *9.5.3.3*

From Example 19.2, the beam-to-slab flexural stiffness ratios α are:

α_f = 13.30 (NS edge beam)

= 16.45 (EW edge beam)

= 3.16 (NS interior beam)

= 3.98 (EW interior beam)

Since all $\alpha_f > 2.0$ (see Fig. 8-2), Eq. (9-13) will control. Therefore,

$$h = \frac{\ell_n (0.8 + f_y / 200{,}000)}{36 + 9\beta}$$ *Eq. (9-12)*

$$= \frac{246\,(0.8 + 60{,}000/200{,}000)}{36 + 9\,(1.28)} = 5.7 \text{ in.}$$

where ℓ_n = clear span in long direction = 20.5 ft = 246 in.

$$\beta = \frac{\text{clear span in long direction}}{\text{clear span in short direction}} = \frac{20.5}{16.0} = 1.28$$

Use 6 in. slab thickness.

2. Frame members of equivalent frame.

Determine moment distribution constants and fixed-end moment coefficients for the equivalent frame members. The moment distribution procedure will be used to analyze the partial frame for vertical loading. Stiffness factors k, carry-over factors COF, and fixed-end moment factors FEM for the slab-beams and column members are determined using the tables of Appendix 20-A. These calculations are shown here.

a. Slab-beams, flexural stiffness at both ends K_{sb}:

$$\frac{c_{N1}}{\ell_1} = \frac{18}{17.5 \times 12} = 0.0857 \approx 0.1$$

$$\frac{c_{N2}}{\ell_2} = \frac{18}{22 \times 12} = 0.0682$$

Referring to Table A1, Appendix 20A,

$$K_{sb} = \frac{4.11 E_c I_{sb}}{\ell_1} = 4.11 \times 25{,}387 E_c/(17.5 \times 12) = 497 E_c$$

where I_{sb} is the moment of inertia of slab-beam section shown in Fig. 20-16 and computed with the aid of Fig. 20-21 at the end of this Example.

$I_{sb} = 2.72\,(14 \times 20^3)/12 = 25{,}387 \text{ in.}^4$

Carry-over factor COF = 0.507

Fixed-end moment, FEM = $0.0842 q_u \ell_2 \ell_1^2$

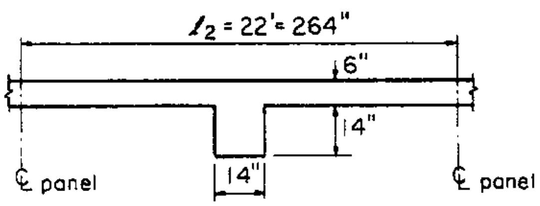


Figure 20-16 Cross-Section of Slab-Beam

b. Column members, flexural stiffness K_c:

$t_a = 17$ in, $t_b = 3$ in., $t_a/t_b = 5.67$

$H = 12$ ft $= 144$ in., $H_c = 144 - 17 - 3 = 124$ in.

$H/H_c = 1.16$ for interior columns

$t_a = 24$ in., $t_b = 3$ in., $t_a/t_b = 8.0$

$H = 12$ ft $= 144$ in., $H_c = 144 - 24 - 3 = 117$ in.

$H/H_c = 1.23$ for exterior columns

Referring to Table A7, Appendix 20A,

For interior columns:

$$K_{ct} = \frac{6.82E_cI_c}{\ell_c} = \frac{6.82 \times 8748E_c}{144} = 414E_c$$

$$K_{cb} = \frac{4.99E_cI_c}{\ell_c} = \frac{4.99 \times 8748E_c}{144} = 303E_c$$

For exterior columns:

$$K_{ct} = \frac{8.57E_cI_c}{\ell_c} = \frac{8.57 \times 8748E_c}{144} = 512E_c$$

$$K_{cb} = \frac{5.31E_cI_c}{\ell_c} = \frac{5.31 \times 8748E_c}{144} = 323E_c$$

where $I_c = \frac{(c)^4}{12} = \frac{(18)^4}{12} = 8{,}748 \text{ in.}^4$

$\ell_c = 12$ ft $= 144$ in.

c. Torsional members, torsional stiffness K_t:

$$K_t = \frac{9E_cC}{\ell_2 (1 - c_2/\ell_2)^3}$$ *R13.7.5*

where $C = \Sigma(1 - 0.63\, x/y)(x^3y/3)$ *13.0*

For interior columns:

$K_t = 9E_c \times 11{,}698/[264\,(0.932)^3] = 493E_c$

where $1 - \dfrac{c_2}{\ell_2} = 1 - \dfrac{18}{(22 \times 12)} = 0.932$

C is taken as the larger value computed with the aid of Table 19-2 for the torsional member shown in Fig. 20-17.

x_1 = 14 in. x_2 = 6 in. y_1 = 14 in. y_2 = 42 in. C_1 = 4,738 C_2 = 2,752	x_1 = 14 in. x_2 = 6 in. y_1 = 20 in. y_2 = 14 in. C_1 = 10,226 C_2 = 736
ΣC = 4,738 + 2,752 = 7,490 in.^{4}	ΣC = 10,226 + 736 × 2 = 11,698 in.^{4}

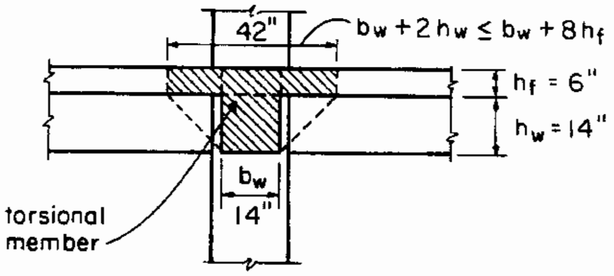


Figure 20-17 Attached Torsional Member at Interior Column

For exterior columns:

$K_t = 9E_c \times 17{,}868/[264\,(0.932)^3] = 752E_c$

where C is taken as the larger value computed with the aid of Table 19-2 for the torsional member shown in Fig. 20-18.

x_1 = 14 in. x_2 = 6 in. y_1 = 21 in. y_2 = 35 in. C_1 = 11,141 C_2 = 2,248	x_1 = 14 in. x_2 = 6 in. y_1 = 27 in. y_2 = 21 in. C_1 = 16,628 C_2 = 1,240
ΣC = 11,141 + 2,248 = 13,389 in.^{4}	ΣC = 16,628 + 1,240 = 17,868 in.^{4}

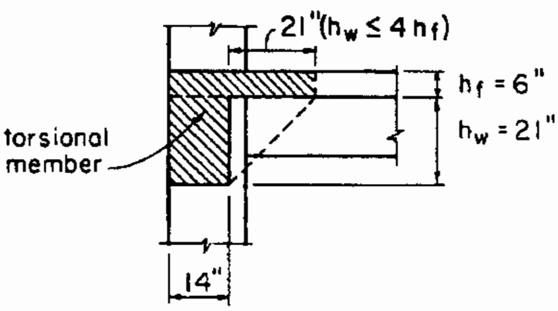


Figure 20-18 Attached Torsional Member at Exterior Column

d. Increased torsional stiffness K_{ta} due to parallel beams:

For interior columns:

$$K_{ta} = \frac{K_t I_{sb}}{I_s} = \frac{493E_c \times 25{,}387}{4{,}752} = 2{,}634E_c$$

For exterior columns:

$$K_{ta} = \frac{752E_c \times 25{,}387}{4{,}752} = 4{,}017E_c$$

where I_s = moment of inertia of slab-section shown in Fig. 20-19.

$= 264\,(6)^3/12 = 4{,}752$ in.^{4}

I_{sb} = moment of inertia of full T-section shown in Fig. 20-19 and computed with the aid of Fig. 20-21

$= 2.72\,(14 \times 20^3/12) = 25{,}387$ in.^{4}

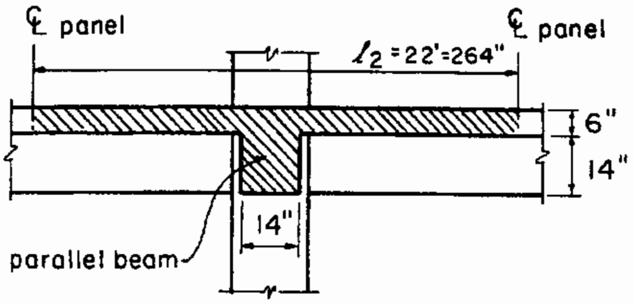


Figure 20-19 Slab-Beam in the Direction of Analysis

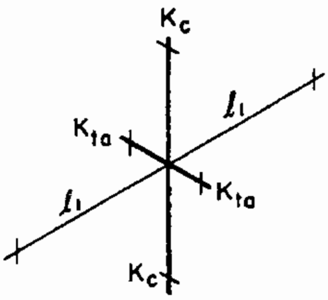
e. Equivalent column stiffness, K_{ec}:

$$K_{ec} = \frac{\Sigma K_c \times \Sigma K_{ta}}{\Sigma K_c + \Sigma K_{ta}}$$

where ΣK_{ta} is for two torsional members, one on each side of column, and ΣK_c is for the upper and lower columns at the slab-beam joint of an intermediate floor.

For interior columns:

$$K_{ec} = \frac{(303E_c + 414E_c)(2 \times 2{,}634E_c)}{(303E_c + 414E_c) + (2 \times 2{,}634E_c)} = 631E_c$$



For exterior columns:

$$K_{ec} = \frac{(323E_c + 521E_c)(2 \times 4{,}017E_c)}{(323E_c + 521E_c) + (2 \times 4{,}017E_c)} = 764E_c$$

f. Slab-beam joint distribution factors DF:

At exterior joint:

$$DF = \frac{497E_c}{(497E_c + 764E_c)} = 0.394$$

At interior joint:

$$DF = \frac{497E_c}{(497E_c + 497E_c + 631E_c)} = 0.306$$

COF for slab-beam = 0.507

3. Partial frame analysis of equivalent frame.

Determine maximum negative and positive moments for the slab-beams using the moment distribution method.

With a service live-to-dead load ratio:

$$\frac{L}{D} = \frac{100}{75} = 1.33 > \frac{3}{4}$$

the frame will be analyzed for five loading conditions with pattern loading and partial live load as allowed by 13.7.6.3 (see Fig. 20-9 for an illustration of the five load patterns considered). *13.7.6.3*

a. Factored loads and fixed-end moments:

Factored dead load, $q_{Du} = 1.2\,(75 + 9.3) = 101$ psf

$\left(\frac{14 \times 14}{144} \times \frac{150}{22} = 9.3 \text{ psf is weight of beam stem per foot divided by } \ell_2\right)$

Factored live load, $q_{Lu} = 1.6\,(100) = 160$ psf

Factored load, $q_u = q_{Du} + q_{Lu} = 261$ psf

FEM for slab-beams $= m_{NF} q_u \ell_2 \ell_1^2$ (Table A1, Appendix 20A)

FEM due to $q_{Du} + q_{Lu}$ $= 0.0842\,(0.261 \times 22)\,17.5^2 = 148.1$ ft-kips