Analysis of time delay difference due to parametric mismatch in matched filter channels

Guillermo Stuarts Member, IEEE, Pedro Julián Senior Member, IEEE

Abstract—In this work, the time delay difference between the outputs of matched filters for a common input is analyzed. The effect of parametric mismatch in the differential time delay is estimated, simulated and measured.

Index Terms—Mismatch, matched filter, time delay difference.

I. INTRODUCTION

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HE estimation of the propagation delay difference between spatially separated sensors plays a key role in many source localization and target tracking applications, such as communications, acoustics, geophysics, sonar systems and sensor networks [1]-[4]. Signal conditioning and filtering is an essential and unavoidable part in all them, at the very first stage of the analog front-end. The use of a filter into the signal path introduces a phase shift that depends on the filter parameters. A slight difference in the filter parameters of the two channels creates a frequency dependent phase difference, that will be interpreted by the processing stage as a delay difference produced by a movement of the target source [5].

This work focuses on the development of expressions that will allow an estimation of the time delay difference between the outputs of two filters, given a common input and parametric mismatch between them.

II. PHASE DIFFERENCE IN SLIGHTLY MISMATCHED CHANNELS

A. Single pole linear system with one frequency tone input

To begin the analysis it is illustrative to explain the situation of a single order low pass filter. In such case, the system has the form:

\[ H(j\omega) = \frac{a}{j\omega + b} \]  

Therefore, the phase is given by

\[ \phi = \tan^{-1}(\omega/b) \]  

Now, let us consider two systems, of the form (1), with a common sinusoidal input, namely \[ u(t) = U \times \sin(\omega t) \], where one of them suffers a variation in the value of the pole, changing from \( b \) to \( b + \Delta b \), i.e

\[ H^1(j\omega) = \frac{a}{j\omega + b} \]  
\[ H^2(j\omega) = \frac{a}{j(\omega + b + \Delta b)} \]  

\[ \phi = \tan^{-1}(\omega/b) \]  
\[ \phi = \tan^{-1}(\omega/(\omega + b + \Delta b)) \]  

The differential time delay (DTD) corresponding to this phase delay is:

\[ \delta(\omega) = (\phi^1(\omega) - \phi^2(\omega))/\omega = \Delta b/(\omega^2 + b^2) \]  

Figure 2 shows a plot of \( \delta(\omega) \) versus frequency for three different values of \( b = \{1, 10, 100\} \). In the three cases, \( \Delta b \) was chosen as a 5% variation on the nominal cut-off frequency value, i.e., \( \Delta b = 0.05 \times b \). It can be clearly appreciated that a maximum of value \( \Delta b/b^2 \) occurs at low frequencies and reduces to zero as frequency increases.

The analysis can be repeated for a single pole high-pass filter, to find that it has exactly the same DTD characteristics as a low pass filter.

B. Higher order filters

If we consider more general systems of the form:

\[ H(s) = \prod_{i=1}^{N} \frac{m_i s^2 + c_i s + d_i}{n_i s^2 + a_i s + b_i} = K \prod_{i=1}^{N} \frac{s - z_i}{s - p_i} \]  

Fig. 1: Setup considered with a nominal system, and a perturbed system.

Accordingly, the outputs will differ in phase. Actually, both outputs will be of the following form:

\[ y^1(t) = || H^1(\omega) || U \times \sin(\omega t + \phi^1(\omega)) \]  
\[ y^2(t) = || H^2(\omega) || U \times \sin(\omega t + \phi^2(\omega)) \]  

The electronic design will try to minimize the difference between the two filters, so it makes sense to assume that \( \Delta b \ll b \) and perform a linearization of \( y^2 \) as a function of \( b \). Considering a variation \( \Delta b \) around the point \( b \), we obtain

\[ \phi^2(\omega) = \tan^{-1}(\omega/b) + (\omega/(\omega^2 + b^2)) \times \Delta b \]  

The authors are with the Department of Electrical and Computer Engineering, Universidad Nacional del Sur (UNS), Instituto de Investigaciones en Ingeniería Eléctrica (IIIE), UNS-CONICET, and Laboratorio de Micro y Nano Electrónica, UNS-Comisión de Investigaciones Científicas (CIC) of Bs. As., Bahía Blanca, Argentina, e-mail: gstuarts@uns.edu.ar.
different amount, and the two systems produce in general two

differences, every frequency component is delayed by a dif-

erent delay, then we obtain an expression for the

delay, and the two systems have a parametric

correlation between two outputs of

The system is a function of frequency for a single-order

Time Delay [s]

Fig. 2: DTD as a function of frequency for a single-order low-

to the linear system contains more than one

Then, we define the DTD between the two signals as

The choice of (13) is based on the widespread use of this

An example of a second order low pass filter is shown in

0.6

Fig. 3.

An example of a second order low pass filter is shown in

0.6

of the poles frequencies, then we obtain an expression for the

DTD that generalizes expression (8)

An example of a second order low pass filter is shown in

0.6

C. Measuring time delay with more frequency components

If the input to the linear system contains more than one

frequency component, and the two systems have a parametric

difference, every frequency component is delayed by a dif-

erent amount, and the two systems produce in general two
different signals. This means that it is not possible to find
out a unique time delay \( \tau \) such that one of the outputs can

be delayed to match the other. Therefore, it is necessary to

introduce a different criteria for time delay.

In this work, we propose the correlation as the tool to define

the time delay between two signals that are different. Given

two signals \( s_1(t), s_2(t) \), the correlation is defined as:

Then, we define the DTD between the two signals as

\[ \tau = \arg \max R_{s_1,s_2}(\tau) \] (13)

The analysis of the differential TD between a pair of signals with

two frequency tones is relevant cause it provides the basis to

analyze the more general case when both signals have multiple

frequency components. Without loss of generality, for this case

we will be considering a situation as illustrated in Fig. 1. If

the input has a description

\[ u(t) = U_1 \sin(\omega_1 t) + U_2 \sin(\omega_2 t) \] (14)

then the two outputs \( y^1(t) \) and \( y^2(t) \) will be of the form:

\[ y^1(t) = \| H^1(\omega_1) \| U_1 \sin(\omega_1(t + \delta^1(\omega_1))) \]

\[ + \| H^1(\omega_2) \| U_2 \sin(\omega_2(t + \delta^1(\omega_2))) \]

\[ y^2(t) = \| H^2(\omega_1) \| U_1 \sin(\omega_1(t + \delta^2(\omega_1))) \]

\[ + \| H^2(\omega_2) \| U_2 \sin(\omega_2(t + \delta^2(\omega_2))) \]

In the following we will consider that the parametric vari-

ation is small enough such that \( \| H^i(\omega_i) \| \approx \| H^j(\omega_j) \| \),

for \( i = 1, 2 \). In addition, and without loss of generality, we will

consider \( y^i(t) \) as reference, and assume that \( \delta^i(\omega_i) = 0 \), for

\( i = 1, 2 \), so that the individual DTD between the tones, is given

by \( \delta^i = \delta^2(\omega_i) \) and \( \delta^2 = \delta^2(\omega_2) \) \(^1\). Having introduced

these considerations, we can formulate the following theorem

that states the result of the correlation between two outputs of

the above mentioned form.

**Theorem 1:** Let

\[ y_1(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) \] (15)

and

\[ y_2(t) = a_1 \sin(\omega_1(t + \delta^{(1)})) + a_2 \sin(\omega_2(t + \delta^{(2)})) \] (16)

two signals with frequencies \( \omega_1, \omega_2 \) and individual differential

delays \( \delta^{(1)}, \delta^{(2)} \), the correlation between them is given by

\[ R(\tau) = A_1 \cos(\omega_1(\delta^{(1)} + \tau)) + A_2 \cos(\omega_2(\delta^{(2)} + \tau)) \] (17)

\(^1\)This notation is used for the sake of conciseness
Some simple algebraic manipulation yields the value for $\tau$ where

$$
A_1 = a_1^2 \left( \frac{\omega_1 t}{2} - \frac{\sin(2\omega_1 t)}{4} \right)
$$

(18)

$$
A_2 = a_2^2 \left( \frac{\omega_2 t}{2} - \frac{\sin(2\omega_2 t)}{4} \right)
$$

(19)

Proof: See The Appendix.

Using the results of Theorem 1, it is possible to find the DTD as the time $\tau$ that maximizes (17). In fact, $\tau$ is obtained from the solution of:

$$
\frac{\partial R}{\partial \tau} = 0
$$

(20)

whose solution is given by:

$$
A_1\sin(\omega_1(\delta^{(1)} + \tau)) = -A_2\sin(\omega_2(\delta^{(2)} + \tau))
$$

(21)

Two cases can be considered here. Let us first examine the case where each individual delay and the solution delay are small compared to the signal frequencies, i.e., $\omega_i(\tau + \delta^{(i)}) \ll 1$ for $i = 1, 2$.

1) Individual DTD small compared to $1/\omega_i$, $i = 1, 2$: In this case, we can approximate $\sin(\omega_i(\tau + \delta^{(i)})) \approx \omega_i(\tau + \delta^{(i)})$ so that (21) reduces to:

$$
A_1 \frac{\omega_1(\delta^{(1)} + \tau)}{\omega_1} = -A_2 \frac{\omega_2(\delta^{(2)} + \tau)}{\omega_2}
$$

(22)

After considering that $t$ is large enough so that $\omega_1 t/2 \gg \sin(2\omega_1 t)/4$ and $\omega_2 t/2 \gg \sin(2\omega_2 t)/4$, (18) and (19) can be written as $A_1 = a_1^2 \omega_1 t/2$ and $A_2 = a_2^2 \omega_2 t/2$ respectively. Some simple algebraic manipulation yields the value for $\tau$:

$$
\tau = a\delta^{(1)} + (1 - a)\delta^{(2)}
$$

(23)

where

$$
a = \frac{\omega_1 a_1^2}{\omega_1 a_1^2 + \omega_2 a_2^2}
$$

(24)

$$
1 - a = \frac{\omega_2 a_2^2}{\omega_1 a_1^2 + \omega_2 a_2^2}
$$

(25)

Several interesting conclusions can be drawn from (23). First of all, notice that the composite DTD is the convex combination of the two individual delays, thus, the DTD is always going to be an intermediate value, i.e., a value in the set

$$
A = [\min(\delta^{(1)}, \delta^{(2)}), \max(\delta^{(1)}, \delta^{(2)})].
$$

(26)

Secondly, notice from (25) that whether the composite DTD is closer to one delay or the other is dependent on the frequency and the square of the amplitude. Therefore, the relative amplitude of one tone versus the other has more influence on the composite DTD than the relative values of the frequencies.

Thirdly, notice that if the amplitudes of the tones are equivalent, then the composite DTD is closer to the individual DTD of the higher frequency signal.

The following examples illustrate these points.

Example 1: Let us consider two low-pass filters of the form (3) and (4), with $a = 1$, $b = 1$, $\Delta b = b/100$ and a signal composed by the sum of two tones $u_1(t) = \sin(\omega_1 t)$, $u_2(t) = \sin(\omega_2 t)$, where $\omega_1 = 0.1$ is kept constant and $\omega_2$ is varied over the range $[10^{-2}, 10^{3}]$. The signals at the output are of the form (15) and (16). Signal $u_1(t)$ is low frequency, so $a_1 \approx 1$, whereas $a_2 = || H(j\omega_2) ||$. According to (8), for a low pass filter, the individual DTD are of the form $\delta^{(i)} = \Delta b/(\omega^2 + b^2)$. As the first tone is fixed, $\delta^{(1)} = \Delta b/(0.1^2 + b^2)$ is constant, whereas $\delta^{(2)} = \Delta b/(\omega_2^2 + b^2)$. Figure 4 shows the individual DTD and the composite DTD. It can be appreciated that at low $\omega_2$ frequencies, both amplitudes are close to one, the individual DTD satisfy $\delta^{(2)} \approx \delta^{(1)} \approx \Delta b/b$ so that the composite DTD is also constant and equal to $\tau \approx \Delta b/b$. In the mid-frequency range of $\omega_2$ we can see that the magnitude has not yet fallen appreciably, but the frequency has increased, therefore, $\tau$ is close to $\delta^{(2)}$. At higher $\omega_2$ frequencies, the amplitude of the tone drops significantly, and due to the quadratic dependence on the amplitude, the composite DTD $\tau$ tends to $\delta^{(1)}$. For this case, $\omega_1(\tau + \delta^{(1)}) < 0.1 \times 2 \times 0.01 \ll 1$ so the assumption on $\omega_1$ is satisfied; for $\omega_2$, $\omega_2(\tau + \delta^{(2)})$ is less than 1 up to $\omega_2 = 100$. As we will see in the next subsection, the effect of $\delta^{(2)}$ at higher frequencies does not affect the composite DTD.

Example 2: Let us consider now two high-pass filters of the form $|| H_1(\omega) || = a j \omega/(j \omega + b)$, and $|| H_2(\omega) || = a j \omega/(j \omega + b + \Delta b)$, with $b = 10$, $\Delta b = -0.1 \times b$ and two signals of frequencies $\omega_1 = 1$, $\omega_2 \in \{10^{-2}, 10^{3}\}$. In this case, $a_i = \omega_i/\sqrt{\omega_i^2 + b^2}$, $i = 1, 2$, so that the low frequency term is $a_1 \approx \omega_1/b$ and the high frequency term is $a_2 \approx 1$. The individual DTD $\delta^{(1)}$ and $\delta^{(2)}$ are as in Example 1. Figure 5 shows the individual and composite DTD as a function of frequency $\omega_2$. It can be seen that at low frequencies $\tau$ is closer to $\delta^{(1)}$, and as frequencies increase, it approaches $\delta^{(2)}$.

2) Individual DTD large compared to $1/\omega_i$: In this case, we will consider the situation where one of the frequencies, for example, $\omega^{(2)}$ is high, so that $\omega_2(\delta^{(2)} + \tau) \gg 1$ while the other still satisfies $\omega_1(\delta^{(1)} + \tau) \ll 1$. In this case, the first term in (17) can be Taylor approximated as a constant term, in a neighborhood of $\tau = -\delta_1$ where it reaches its maximum.
value, i.e., $A_1 \frac{\cos(\omega_1 (\delta_1 + \tau))}{\omega_1} \approx A_1/\omega_1$, therefore, $R(\tau)$ can be written as:

$$R(\tau) = \frac{A_1}{\omega_1} + A_2 \frac{\cos(\omega_2 (\delta_2 + \tau))}{\omega_2}$$

(27)

The second term in Eq. (27) is periodic, and has maximum values on the points

$$(-\delta_2 + 2k\pi/\omega_2, k \in \mathbb{Z}).$$

(28)

Since $R(\tau)$ has been reduced to the sum of a constant term plus a cosine function, then, it is clear that its maximum will coincide with the maximum of the cosine function that is closer to $-\delta_1$. If $k$ is allowed to be a real number, the solution of (28) coincident at $\tau = -\delta_1$ would be $k = \omega_2 (\delta_2 - \delta_1)/(2\pi)$. Therefore the solution is given by

$$\tau = -\delta_2 + 2k^*\pi/\omega_2$$

$$k^* = \text{round}(\omega_2 (\delta_2 - \delta_1)/(2\pi))$$

(29)

Figure 6 shows a plot of the composite DTD, when there are two signals, one with a fixed frequency $\omega_1$ and a DTD $\delta_1 = 0.05$, and the other with a frequency $\omega_2 \in [10^{-2}, 10^{3}]\text{rad/s}$ and a DTD $\delta_2 = 0.02$. The solid line is the exact numerical result, and the dotted line is the result of the proposed approximations.
Fig. 8: DTD calculated and estimated from the measurements of single-order high-pass filters.

B. Low-Pass RC Filters

The low-pass filters in Figure 9 presented cut-off frequencies of $f_{c1} = 20988$ Hz and $f_{c2} = 19827$ Hz. The numerical calculation of the DTD and the estimation from the correlation analysis are shown in Figure 10.

Fig. 9: Single order low-pass filter used to perform the measurements.

Fig. 10: DTD calculated and estimated from the measurements of single order low-pass filters.

C. Band-Pass RC Filters

Cascading the high-pass and low-pass filters from the previous subsections, two bandpass filters were implemented and the results are shown in Figure 11.

Fig. 11: DTD calculated and estimated from the measurements of band-pass filters implemented as the cascade of the previous high-pass and low-pass filters.

D. Multitone input

Expressions (23) and (29) provide an estimation of the DTD when the input is a combination of two tones. In order to validate those expressions, the filter setup of Figure 12 was implemented. The input signal was constructed as the sum of a constant frequency tone $\omega_1 = 500$ Hz and a variable frequency tone $\omega_2$.

Firstly, two Butterworth second-order low-pass filters were implemented with the SIM965 filters, with cut-off frequencies of 1 KHz and 950 Hz. In this case it will hold that the individual DTD will be small compared to $1/\omega_i$, $i = 1, 2$ and so expression (23) was used to estimate the composite DTD. Figure 13 shows the results of the measurements.

Fig. 12: Filter set-up used to perform the measurements.
Secondly, two second-order high-pass Butterworth filters were implemented with the SIM965 filters, with cut-off frequencies of 500 Hz and 450 Hz. In this case individual DTD will be large compared to 1/\(\omega_i\), \(i = 1, 2\) and so expression (29) was used to estimate the composite DTD. Figure 14 shows the results of the measurements.

\[ R(\tau) = y_1(t) \ast y_2(t) = \int [a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t)] \times [a_1 \sin(\omega_1 (t + \delta_1 + \tau)) + a_2 \sin(\omega_2 (t + \delta_2 + \tau))] \, dt \]

Let \(\tau_1 = \delta_1 + \tau\); \(\tau_2 = \delta_2 + \tau\) then:

\[ R(\tau) = \int [a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t)] \times [a_1 \sin(\omega_1 (t + \tau_1)) + a_2 \sin(\omega_2 (t + \tau_2))] \, dt = \]

\[ = \int [a_1^2 \sin(\omega_1 t) \sin(\omega_1 (t + \tau_1))] \, dt + \]

\[ + \int [a_2^2 \sin(\omega_2 t) \sin(\omega_2 (t + \tau_2))] \, dt + \]

\[ + \int [a_1 a_2 \sin(\omega_1 t) \sin(\omega_2 (t + \tau_2))] \, dt + \]

\[ + \int [a_1 a_2 \sin(\omega_2 t) \sin(\omega_1 (t + \tau_1))] \, dt \]

Since:

\[ \sin(\omega(t + \tau)) = \sin(\omega t) \cos(\omega \tau) + \cos(\omega t) \sin(\omega \tau) \]

Then:

\[ \int [a_1^2 \sin(\omega_1 t) \sin(\omega_1 (t + \tau_1))] \, dt = \]

\[ = \int a_1^2 \sin(\omega_1 t) \sin(\omega_1 (t + \tau_1)) \, dt \]

\[ = a_1^2 \cos(\omega_1 \tau_1) \int \sin^2(\omega_1 t) \, dt + \]

\[ + a_1^2 \sin(\omega_1 \tau_1) \int \sin(\omega_1 t) \cos(\omega_1 t) \, dt \]

\[ = a_1^2 \cos(\omega_1 \tau_1) \int \sin^2(\omega_1 t) \, dt + a_1^2 \sin(\omega_1 \tau_1) \int \sin(2\omega_1 t) \, dt \]

\[ \int [a_2^2 \sin(\omega_2 t) \sin(\omega_2 (t + \tau_2))] \, dt = \]

\[ = \int a_2^2 \sin(\omega_2 t) \sin(\omega_2 (t + \tau_2)) \, dt + \]

\[ + \cos(\omega_2 t) \sin(\omega_2 \tau_2) \, dt \]

\[ = a_2^2 \cos(\omega_2 \tau_2) \int \sin^2(\omega_2 t) \, dt + a_2^2 \sin(\omega_2 \tau_2) \int \sin(2\omega_2 t) \, dt \]

**IV. CONCLUSIONS**

Several expressions were developed for the estimation of time delay as a function of filter parameters. The case of a single tone going through a filter was analysed and expression (11) was proposed as an estimation for the differential time delay between the outputs of two generalized filters of any order. Experimental and simulated results show good matching with the actual numerical calculation.

Finally, expressions (23) and (29) were obtained for the estimation of the DTD when the input is a multi-tone signal. Correlation between the outputs was selected as the tool to derive such expressions and again a set of measurements were performed on filters of different type and order to validate the results.
Replacing:

\[ \int [a_1 a_2 \sin(\omega_1 t) \sin(\omega_2 (t + \tau_2))] dt \]

Since:

\[ \sin(\omega_1 t) \sin(\omega_2 t) = \frac{1}{2} [\cos((\omega_1 - \omega_2) t) - \cos((\omega_1 + \omega_2) t)] \]

\[ \sin(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2} [\sin((\omega_1 + \omega_2) t) + \sin((\omega_1 - \omega_2) t)] \]

Removing the terms with zero mean value:

\[ \int [a_1 a_2 \sin(\omega_1 t) \sin(\omega_2 (t + \tau_2))] dt = \]

\[ + a_1 a_2 \cos(\omega_2 \tau_2) \int [\sin((\omega_1 - \omega_2) t) - \cos((\omega_1 + \omega_2) t)] dt \]

\[ + a_1 a_2 \sin(\omega_1 \tau_1) \int [\cos((\omega_2 - \omega_1) t) - \cos((\omega_2 + \omega_1) t)] dt \]

Also:

\[ \int [a_1 a_2 \sin(\omega_2 t) \sin(\omega_1 (t + \tau_1))] dt = \]

\[ + a_1 a_2 \cos(\omega_1 \tau_1) \int [\cos((\omega_2 - \omega_1) t) - \cos((\omega_2 + \omega_1) t)] dt \]

Removing the terms with zero mean value:

\[ R(\tau) = a_1^2 \int 2 \sin^2(\omega_1 t) dt + \]

\[ + a_2^2 \cos(\omega_2 \tau_2) \int 2 \sin^2(\omega_2 t) dt \]

Since: \[ \sin^2(\omega t) = \frac{1}{2} - \frac{\cos(2\omega t)}{2} \]

\[ R(\tau) = a_1^2 \cos(\omega_1 \tau_1) \int \left[ \frac{1}{2} - \frac{\cos(2\omega_1 t)}{2} \right] dt \]

\[ + a_2^2 \cos(\omega_2 \tau_2) \int \left[ \frac{1}{2} - \frac{\cos(2\omega_2 t)}{2} \right] dt \]

\[ = a_1^2 \cos(\omega_1 \tau_1) \left[ \frac{t}{2} - \frac{\sin(2\omega_1 t)}{4\omega_1} \right] \]

\[ + a_2^2 \cos(\omega_2 \tau_2) \left[ \frac{t}{2} - \frac{\sin(2\omega_2 t)}{4\omega_2} \right] \]

Replacing: \[ \tau_1 = \delta_1 + \tau; \tau_2 = \delta_2 + \tau \]

\[ R(\tau) = a_1^2 \left[ \frac{t}{2} - \frac{\sin(2\omega_1 t)}{4\omega_1} \right] \cos(\omega_1 (\delta_1 + \tau)) + \]

\[ + a_2^2 \left[ \frac{t}{2} - \frac{\sin(2\omega_2 t)}{4\omega_2} \right] \cos(\omega_2 (\delta_2 + \tau)) \]

\[ = a_1^2 \left[ \frac{t \omega_1}{2} - \frac{\sin(2\omega_1 t)}{4} \right] \cos(\omega_1 (\delta_1 + \tau)) + \]

\[ + a_2^2 \left[ \frac{t \omega_2}{2} - \frac{\sin(2\omega_2 t)}{4} \right] \cos(\omega_2 (\delta_2 + \tau)) \]

Let us define:

\[ A_1 = a_1^2 \left[ \frac{t \omega_1}{2} - \frac{\sin(2\omega_1 t)}{4} \right] \]

\[ A_2 = a_2^2 \left[ \frac{t \omega_2}{2} - \frac{\sin(2\omega_2 t)}{4} \right] \]

Then:

\[ \hat{R}(\tau) = A_1 \frac{\cos(\omega_1 (\delta_1 + \tau))}{\omega_1} + A_2 \frac{\cos(\omega_2 (\delta_2 + \tau))}{\omega_2} \]

\[ \square \]

REFERENCES