

## **Resistance Factors for Settlement Design**

*by Gordon A. Fenton<sup>1</sup>, D. V. Griffiths<sup>2</sup>, and W. Cavers<sup>3</sup>*

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### **Abstract**

In order to control serviceability problems arising from excessive settlement of shallow footings, geotechnical design codes generally include specifications regarding maximum settlement which often govern the footing design. Once the footing has been designed and constructed, the actual settlement it experiences on a real three-dimensional soil mass can be quite different than expected, due to the soil's spatial variability. Because of this generally large variability (compared to other engineering materials such as concrete and steel) and because this particular serviceability limit state often governs the design, it makes sense to consider a reliability-based approach to settlement design. This paper looks in some detail at a Load and Resistance Factor Design (LRFD) approach to limiting footing settlement. In particular, the resistance factors required to achieve a certain level of settlement reliability as a function of soil variability and site investigation intensity are determined analytically using random field theory. Simplified approximate relationships are proposed and tested using simulation via the Random Finite Element Method. It is found that the simplified relationships are validated both by theory and simulation and so can be used to augment the calibration of geotechnical LRFD code provisions with respect to shallow foundation settlement.

**keywords:** reliability-based design, settlement, geotechnical, shallow foundation, random field, probability

## 1. Reliability-Based Geotechnical Design Issues

In an effort to harmonize with structural codes, geotechnical design codes around the world are beginning to migrate towards some form of reliability-based design (RBD). Significant steps in this direction can be found in, for example, Eurocode 7, 2004, Australian Standard AS 4678, 2002, NCHRP Report 507, 2004, and the Canadian Foundation Engineering Manual, 1992. These RBD provisions are most often presented in the form of a Limit States Design (LSD), to define critical failure states, combined with load and resistance factors calibrated to achieve the target reliabilities associated with the various limit states. The use of load and resistance factors is generally referred to as Load and Resistance Factor Design (LRFD).

By and large, the random characteristics of loads, or “actions”, in civil engineering projects, are fairly well known and so load factors are reasonably well established. On the resistance side, for most common structural materials representative tests can easily be performed, and have been, to establish material property distributions that apply with reasonable accuracy anywhere that the material is used. Thus, resistance factors for materials such as concrete, steel, and wood have been known for decades.

Unfortunately, the development of resistance factors for use in geotechnical engineering is much more difficult than for quality-controlled engineering materials, such as concrete, steel, or wood. For example, while the mean strength of a batch of 30 MPa concrete delivered to a site in one city, might differ by 5 to 10% from a batch delivered to a site in a second city, the soil strengths at the two sites may easily differ by orders of magnitude. A significant advantage of designing using quality-controlled materials is that the general form and, in particular, the variance of the material property distribution is well enough accepted by the engineering profession that only a few samples of the material are deemed necessary to ensure design requirements are met. That is, engineers rely on an *a priori* estimate of the material variance which means that testing can be aimed at only ensuring that the mean material resistance is sufficiently high (the number of samples taken is usually far too few to accurately estimate the variance). This is essentially a hypothesis test on the mean with variance known. Using this test to ensure that design requirements are met, combined with the known distributions and resulting codified load and resistance factors, is sufficient to provide a reasonably accurate reliability-based design.

Contrast the knowledge regarding the distribution of, say, concrete with that of soils. In analogy to the above discussion, it would be nice to have a reasonably accurate *a priori* estimate of soil property variance, so that only the mean soil property would have to be determined via a site investigation. Such an *a priori* estimate of variance would involve sampling many sites across the world – some in gravel, some in swamps, some in rock, some in clay, sand, and so on – and then estimating the variance across these samples. This *a priori* variance would be very large and this has two implications; first, more samples would be required to accurately estimate the mean at a site and secondly, and probably more importantly, the resulting reliability-based designs will often be overly conservative and expensive. That is, this ‘worst case’ *a priori* variance for soils would generally be much larger than the actual variance at a single site, which would typically lead to overdesign in order to achieve a certain reliability. Nevertheless, an *a priori* variance for soils would be of some value, particularly in situations where the site investigation is insufficient to estimate the variance, or for preliminary designs. In addition, it is better to start out on the safe side, and refine the design as sufficient information is gathered.

The above argument suggests that in order to achieve efficient reliability-based geotechnical designs, the site investigation must be intensive enough to allow the estimation of both the soil mean and variance – this level of site investigation intensity is typically what is aimed at in modern geotechnical codes, with varying degrees of success (for example, Australian Standard AS 4678, 2002, specifies three different investigation levels, associated with three different reliability levels). To date, however, little guidance is provided on how to determine “characteristic” design values for the soil on the basis of the gathered data, nor on how to use the estimated variance to adjust the design.

Another complicating factor, which is more of a concern in soils than in other quality-controlled materials, is that of spatial variability and its effect on design reliability. Soil properties often vary markedly from point to point and this variability can have quite different importance for different geotechnical issues. For example, footing settlement, which depends on an average property under the footing, is only moderately affected by spatial variability, while slope stability, which involves the path of least resistance, is more strongly affected by spatial variability. In this paper, spatial variability will be simply characterized by a parameter referred to here as the correlation length – small correlation lengths imply more rapidly varying properties, and so on. In order to adequately characterize the probabilistic nature of a soil and arrive at reasonable reliability-based designs, then, three parameters need to be estimated at each site; the mean, variance, and correlation length.

Fortunately, evidence compiled by the authors in the past indicates that a ‘worst case’ correlation length typically exists – this means that, in the absence of sufficient data, this worst case can be used in reliability calculations. It will generally be true that insufficient data are collected at a site to reasonably estimate the correlation length, so the worst case value is appropriate to use (despite the fact that this is somewhat analogous to using the worst case a priori variance discussed above).

Once the random soil at a site has been characterized in some way, the question becomes how should this information be used in a reliability-based design? In this paper, a Limit State Design approach will be considered, where a square footing is placed on a three-dimensional soil mass and the task is to design the footing to have a sufficiently high reliability against excessive settlement. Thus, the limit state in question is a serviceability limit state. In structural design, serviceability limit states are investigated using unfactored loads and resistances. In keeping with this, both the Eurocode 7 (2004) and Australian Standard AS 2159 (1995) specify unit resistance factors for serviceability limit states. The Australian Standard AS 5100.3 (2004) states that “a geotechnical reduction factor need not be applied” for serviceability limit states.

Due to the inherently large variability of soils, however, and because settlement often governs a design, it is the opinion of the authors that properly selected resistance factors should be used for both ultimate and serviceability limit states. The Australian Standard AS 4678 (2002), for example, agrees with this opinion and, in fact, distinguishes between resistance factors for ultimate limit states and serviceability limit states – the factors for the latter are closer to 1.0, reflecting the reduced reliability required for serviceability issues. Although the Canadian Foundation Engineering Manual (3rd Ed., 1992) suggests the use of a “performance factor” (foundation capacity reduction factor) of unity for settlement, it goes on to say “However, in view of the uncertainty and great variability in in situ soil-structure stiffnesses, Meyerhof (1982) has suggested that a performance factor of 0.7 should be used for an adequate reliability of serviceability estimates.”

If resistance factors are to be used, how should they be selected so as to achieve a certain reliability? Statistical methods suggest that the resistance factors should be adjusted until a sufficiently small

fraction of possible realizations of the soil enter the limit state being designed against. Unfortunately, there is only one realization of each site and, since all sites are different, it is difficult to apply statistical methods to this problem. For this reason geotechnical reliability-based code development has largely been accomplished by calibration with past experience as captured in previous codes. This is quite acceptable, since design methodologies have evolved over many years to produce a socially acceptable reliability, and this encapsulated information is very valuable – see, for example, Vick’s (2002) discussion of the value of judgement in engineering.

On the other hand, a reliability-based design code derived purely from deterministic codes cannot be expected to provide the additional economies that a true reliability-based design code could provide, eg. by allowing the specification of the target reliability (lower reliability for less important structures, etc.), or by improving the design as uncertainty is reduced, and so on. To attain this level of control in a reliability-based design code, probabilistic modeling and/or simulation of many possible soil regimes should also be employed to allow the investigation of the effect that certain design parameters have on system reliability. This is an important issue – it means that probabilistic modeling is necessary if reliability-based design codes are to evolve beyond being mirror images of the deterministic codes they derive from. The randomness of soils must be acknowledged and properly accounted for.

This paper presents the results of a study in which a reliability-based settlement design approach is proposed and investigated via simulation using the Random Finite Element Method (RFEM). In particular, the effect of a soil’s spatial variability and site investigation intensity on the resistance factors is quantified. The results of the paper can and should be used to improve and generalize “calibrated” code provisions based purely on past experience.

## **2. Random Finite Element Method (RFEM)**

A specific settlement design problem will be considered here in order to investigate the settlement probability distribution of footings designed against excessive settlement. The problem considered is that of a rigid rough square pad footing founded on the surface of a three-dimensional linearly elastic soil mass underlain by bedrock at depth  $H$ . Although only elastic settlement is specifically considered, the results can include consolidation settlement so long as the combined settlement can be adequately represented using an effective elastic modulus field. To the extent that the elastic modulus itself is a simplified representation of a soil’s inverse compressibility, which is strain level dependent, the extension of the approximation to include consolidation settlement is certainly reasonable, and is as recommended in the Canadian Highway Bridge Design Code Commentary (2000).

The settlement of a rigid footing on a three-dimensional soil mass is estimated using a linear finite element analysis. The mesh selected is 64 elements by 64 elements in plan by 32 elements in depth. Eight-node hexahedral elements, each cubic with side length 0.15 m are used (note that metric units are used in this paper, rather than making it non-dimensional, since footing design will be based on a maximum tolerable settlement which is specified in m) yielding a soil domain of size  $9.6 \times 9.6$  m in plan by 4.8 m in depth. Because the stiffness matrix corresponding to a mesh of size  $64 \times 64 \times 32$  occupies about 4 Gbytes of memory, a preconditioned conjugate gradient iterative solver, which avoids the need to assemble the global stiffness matrix, is employed in the finite element code. A max-norm relative error tolerance of 0.005 is used to determine when the iterative solver has converged to a solution.

The finite element model was tested (see also Griffiths and Fenton ) in the deterministic case (uniform elastic soil properties) to validate its accuracy and was found to be about 20% stiffer (smaller settlements) than that derived analytically (see, eg, Milovic 1992). Using other techniques such as selectively reduced integration, non-conforming elements, and 20-node elements did not significantly affect the discrepancy between these results and Milovic's. The 'problem' is that the finite elements truncate the singular stresses that occur along the edge of a rigid footing, leading to smaller settlements than predicted by theory. In this respect, Seyček (1991) compares real settlements to those predicted by theory and concluded that predicted settlements are usually considerably higher than real settlements. This is because the true stresses measured in the soil near the footing edge are finite and significantly less than the singular stresses predicted by theory. Seyček improves the settlement calculations by reducing the stresses below the footing. Thus, the finite element results included here are apparently closer to actual settlements than those derived analytically, although a detailed comparison to Seyček's has yet to be performed by the authors. However, it is not believed that these possible discrepancies will make a significant difference to the probabilistic results of this paper since the probability of failure (excessive settlement) involves a comparison between deterministic and random predictions arising from the same finite element model, thus cancelling out possible bias.

The rigid footing is assumed to have a rough interface with the underlying soil – no relative slip is permitted – and rotation of the footing is not permitted. Only square footings, of dimension  $B \times B$  are considered, where the required footing width  $B$  is determined during the design phase, to be discussed in the next section. Once the required footing width has been found, the design footing width must be increased to the next larger element boundary – this is because the finite element mesh is fixed and footings must span an integer number of elements. For example, if the required footing width is 2.34 m, and elements have dimension  $\Delta x = \Delta y = 0.15$  m square, then the design footing width must be increased to 2.4 m (since this corresponds to 16 elements, rather than the 15.6 elements that 2.34 m would entail). This corresponds roughly to common design practice, where element dimensions are increased to an easily measured quantity.

Once the design footing width has been found, it must be checked to ensure that it is physically reasonable, both economically and within the finite element model. First of all, there will be some minimum footing size. In this study the footings cannot be less than 4 x 4 elements in size – for one thing loaded areas smaller than this tend to have significant finite element errors, for another, they tend to be too small to construct. For example, if an element size of 0.15 m is used, then the minimum footing size is 0.6 x 0.6 m, which is not very big. French (1999) recommends a lower bound on footing size of 0.6 m and an upper economical bound of 3.7 m. If the design footing width is less than the minimum footing width, it is set equal to the minimum footing width. Secondly, there will be some maximum footing size. A spread footing bigger than about 4 m square would likely be replaced by some other foundation system (piles, mat, or raft). In this program, the maximum footing size is taken to be equal to 2/3 of the finite element mesh width. This limit has been found to result in less than a 1% error relative to the same footing founded on a mesh twice as wide, so boundary conditions are not significantly influencing the results. If the design footing width exceeds the maximum footing width then the probabilistic interpretation becomes somewhat complicated, since a different design solution would presumably be implemented. From the point of view of assessing the reliability of the 'designed' spread footing, it is necessary to decide if this excessively large footing design would correspond to a success, or to a failure. It is assumed in this study that the subsequent design of the alternative foundation would be a success, since it would have its own (high) reliability.

In all the simulations performed in this study, the lower limit on the footing size was never encountered, implying that for the choices of parameters selected in this study, the probability of a design footing being less than  $0.6 \times 0.6$  in dimension was very remote. Similarly, the maximum footing size was not exceeded in any but the most severe parameter case considered (minimum sampling, lowest resistance factor, highest coefficient of variation), where it was only exceeded in 2% of the possible realizations. Thus, the authors were satisfied that the finite element analysis would give reasonably accurate settlement predictions over the entire study.

The soil property of primary interest to settlement is elastic modulus,  $E$ , which is taken to be spatially random and may represent both the initial elastic and consolidation behaviour. Its distribution is assumed to be lognormal for two reasons: the first is that a geometric average tends to a lognormal distribution by the central limit theorem and the effective elastic modulus, as ‘seen’ by a footing, was found to be closely represented by a geometric average in Fenton and Griffiths (2002), and second is that the lognormal distribution is strictly non-negative which is physically reasonable for elastic modulus. The lognormal distribution has two parameters,  $\mu_{\ln E}$  and  $\sigma_{\ln E}$  which can be estimated by the sample mean and sample standard deviation of observations of  $\ln(E)$ . They can also be obtained from the mean and standard deviation of  $E$  using the transformations

$$\begin{aligned} [1a] \quad & \sigma_{\ln E}^2 = \ln(1 + V_E^2) \\ [1b] \quad & \mu_{\ln E} = \ln(\mu_E) - \frac{1}{2}\sigma_{\ln E}^2 \end{aligned}$$

where  $V_E = \sigma_E / \mu_E$  is the coefficient of variation of the elastic modulus field. A Markovian spatial correlation function, which gives the correlation coefficient between log-elastic modulus values at points separated by the lag vector,  $\underline{\tau}$ , is used in this study;

$$[2] \quad \rho_{\ln E}(\underline{\tau}) = \exp \left\{ -\frac{2|\underline{\tau}|}{\theta_{\ln E}} \right\}$$

in which  $\underline{\tau} = \underline{x} - \underline{x}'$  is the vector between spatial points  $\underline{x}$  and  $\underline{x}'$ , and  $|\underline{\tau}|$  is the absolute length of this vector (the lag distance). In this paper, the word ‘correlation’ refers to the correlation coefficient. The results presented here are not particularly sensitive to the choice in functional form of the correlation – the Markov model is popular because of its simplicity. The correlation function decay rate is governed by the so-called correlation length,  $\theta_{\ln E}$ , which, loosely speaking, is the distance over which log-elastic moduli are significantly correlated (when the separation distance  $|\underline{\tau}|$  is greater than  $\theta_{\ln E}$ , the correlation between  $\ln E(\underline{x})$  and  $\ln E(\underline{x}')$  is less than 14%). The correlation structure is assumed to be isotropic in this study which is appropriate for investigating the fundamental stochastic behaviour of settlement. Anisotropic studies are more appropriate for site-specific analyses and for refinements to this study. In any case, anisotropy is not expected to have a large influence on the results of this paper due to the averaging effect of the rigid footing on the properties it ‘sees’ beneath it.

Poisson’s ratio, having only a relatively minor influence on settlement, is assumed to be deterministic and is set at 0.3 in this study.

Realizations of the random elastic modulus field are produced using the Local Average Subdivision (LAS) method (Fenton and Vanmarcke 1990). Specifically, LAS produces a discrete grid of local averages,  $G(\underline{x}_i)$ , of a standard Gaussian random field, having correlation structure given by Eq. (2),

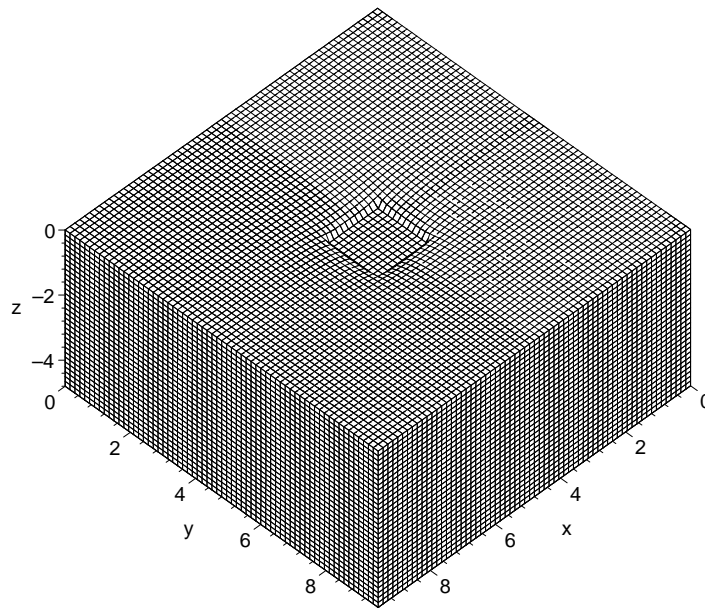
where  $x_i$  are the coordinates of the centroid of the  $i^{\text{th}}$  grid cell. These local averages are then mapped to finite element properties according to

$$[3] \quad E(x_i) = \exp \{ \mu_{\ln E} + \sigma_{\ln E} G(x_i) \}$$

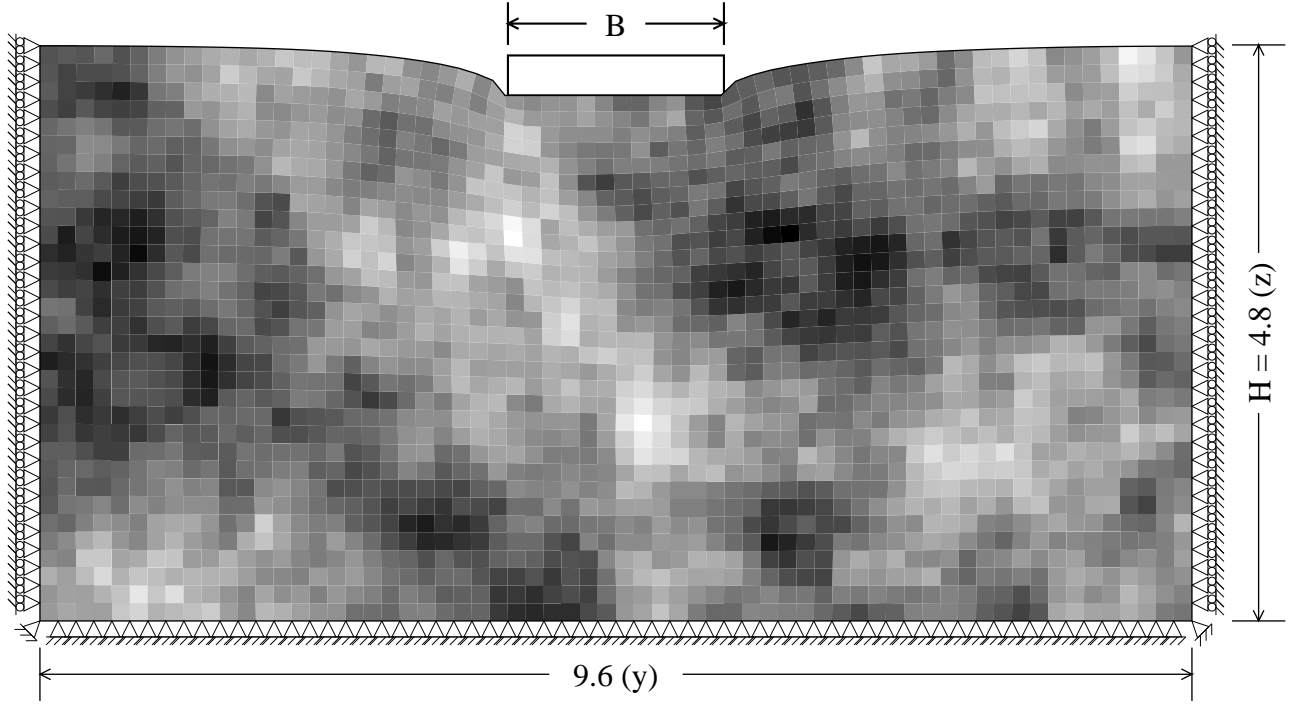
(which assumes that the centroids of the random field cells and the finite elements coincide, as they do in this study).

Much discussion of the relative merits of various methods of representing random fields in finite element analysis has been carried out in recent years (see, for example, Li and Der Kiureghian, 1993). While the spatial averaging discretization of the random field used in this study is just one approach to the problem, it is appealing in the sense that it reflects the simplest idea of the finite element representation of a continuum as well as the way that soil samples are typically taken and tested in practice, ie. as local averages. Regarding the discretization of random fields for use in finite element analysis, Matthies et al. (1997) makes the comment that “One way of making sure that the stochastic field has the required structure is to assume that it is a local averaging process.”, referring to the conversion of a nondifferentiable to a differentiable (smooth) stochastic process. Matthie further goes on to say that the advantage of the local average representation of a random field is that it yields accurate results even for rather coarse meshes.

Figure 1 illustrates the finite element mesh used in the study and Figure 2 shows a cross-section through the soil mass under the footing for a typical realization of the soil’s elastic modulus field. Figure 2 also illustrates the boundary conditions.



**Figure 1.** Finite element mesh with one square footing.



**Figure 2.** Cross-section through a realization of the random soil underlying the footing. Lighter soils are softer.

### 3. Reliability-Based Settlement Design

The goal of this paper is to propose and investigate a reliability-based design methodology for the serviceability limit state of footing settlement. Footing settlement is predicted here using a modified Janbu (1956) relationship, and this is the basis of design used in this paper;

$$[4] \quad \delta_p = u_1 \frac{\hat{q}B}{\hat{E}}$$

where  $\delta_p$  is the predicted footing settlement,  $\hat{q} = \hat{P}/B^2$  is the estimated stress applied to the soil by the estimated load,  $\hat{P}$ , acting over footing area  $B \times B$ ,  $\hat{E}$  is the (possibly drained) estimate of elastic modulus underlying the footing,  $u_1$  is an influence factor which includes the effect of Poisson's ratio ( $\nu = 0.3$  in this study). The estimated load,  $\hat{P}$ , is often a nominal load computed from the supported live and dead loads, while the estimated elastic modulus,  $\hat{E}$ , is usually a cautious estimate of the mean elastic modulus under the footing obtained by taking laboratory samples or by in-situ tests, such as CPT. In terms of the footing load,  $\hat{P}$ , the settlement predictor thus becomes

$$[5] \quad \delta_p = u_1 \frac{\hat{P}}{B\hat{E}}$$

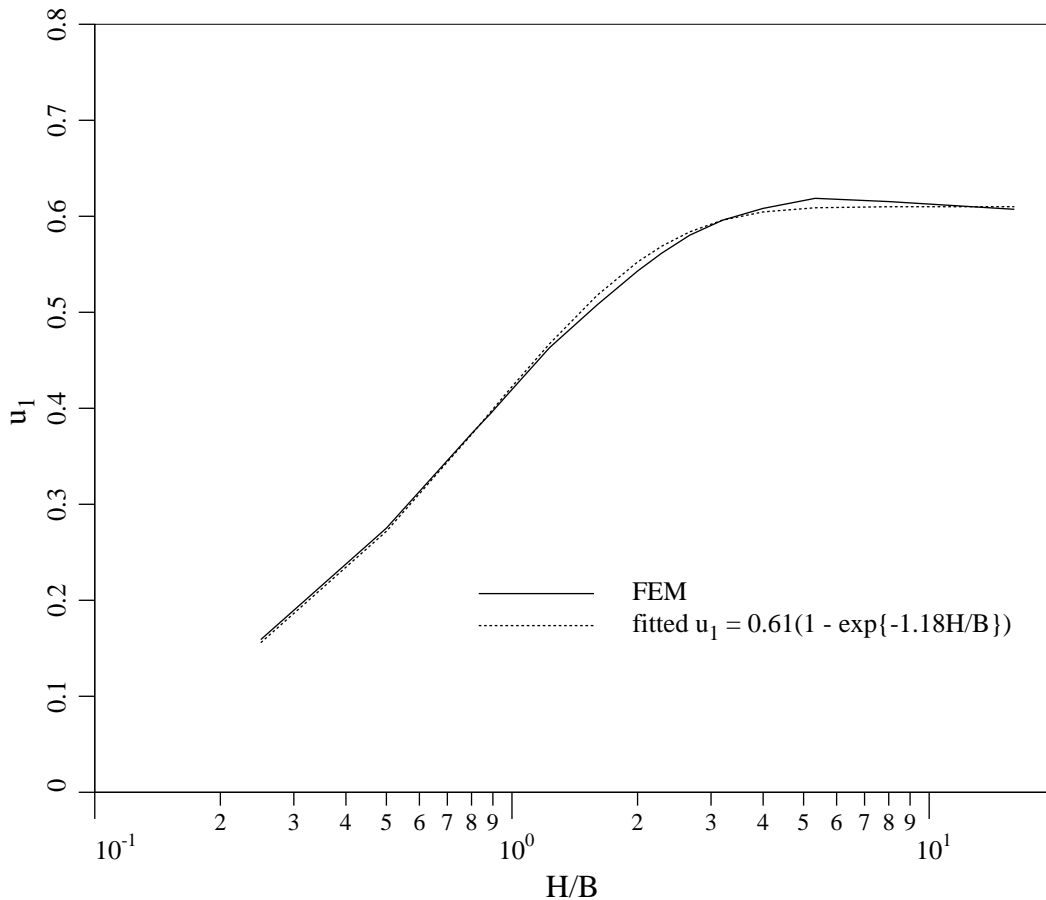
The relationship above is somewhat modified from that given by Janbu (1956) and Christian and Carrier (1978) in that the influence factor,  $u_1$ , is calibrated specifically for a square rough rigid

footing founded on the surface of an elastic soil using the same finite element model which is later used in the Monte Carlo simulations. This is done to remove bias (model) errors and concentrate specifically on the effect of spatial soil variability on required resistance factors. In practice, this means that the resistance factors proposed in this paper are *upper bounds*, appropriate for use when bias and measurement errors are known to be minimal.

The calibration of  $u_1$  is done by computing the deterministic (non-random) settlement of an elastic soil with elastic modulus  $\hat{E}$  and Poisson's ratio  $\nu$  under a square rigid rough footing supporting load  $\hat{P}$  using the finite element program. Once the settlement is obtained, Eq. (5) can be solved for  $u_1$ . Repeating this over a range of  $H/B$  ratios leads to the curve shown in Figure 3. (Note that this deterministic calibration was carried out over a larger range of mesh dimensions than indicated by Figure 1.) A very close approximation to the finite element results is given by the fitted relationship (obtained by consideration of the correct limiting form and by trial-and-error for the coefficients)

$$[6] \quad u_1 = 0.61 \left( 1 - e^{-1.18H/B} \right)$$

which is also shown on Figure 3.



**Figure 3.** Calibration of  $u_1$  using finite element model.

Using Eq. (6) in Eq. (5) gives the following settlement prediction

$$[7] \quad \delta_p = 0.61 \left( 1 - e^{-1.18H/B} \right) \left( \frac{\hat{P}}{B\hat{E}} \right)$$

The reliability-based design goal is to determine the footing width,  $B$ , such that the probability of exceeding a specified tolerable settlement,  $\delta_{max}$ , is acceptably small. That is, to find  $B$  such that

$$[8] \quad \mathbf{P}[\delta > \delta_{max}] = p_f = p_{max}$$

where  $\delta$  is the actual settlement of the footing as placed (which will be considered here to be the same as designed). Design failure is assumed to have occurred if the actual footing settlement,  $\delta$ , exceeds the maximum tolerable settlement,  $\delta_{max}$ . The probability of design failure is  $p_f$  and  $p_{max}$  is the maximum acceptable risk of design failure.

A realization of the footing settlement,  $\delta$ , is determined here using a finite element analysis of a realization of the random soil. For  $u_1$  calibrated to the finite element results,  $\delta$  can also be computed from

$$[9] \quad \delta = u_1 \frac{P}{BE_{eff}}$$

where  $P$  is the actual footing load and  $E_{eff}$  is the effective elastic modulus as seen by the footing (ie, the uniform value of elastic modulus which would produce a settlement identical to the actual footing settlement). Both  $P$  and  $E_{eff}$  are random variables.

One way of achieving the desired design reliability is to introduce a load factor,  $\alpha \geq 1$ , and a resistance factor,  $\phi \leq 1$ , and then finding  $B$ ,  $\alpha$  and  $\phi$  which satisfy both Eq. (8) and Eq. (5) with  $\delta = \delta_{max}$ . In other words, find  $B$  and  $\alpha/\phi$  such that

$$[10] \quad \delta_{max} = u_1 \left( \frac{\alpha\hat{P}}{B\phi\hat{E}} \right)$$

and

$$[11] \quad \mathbf{P} \left[ u_1 \frac{P}{BE_{eff}} > u_1 \left( \frac{\alpha\hat{P}}{B\phi\hat{E}} \right) \right] = p_{max}$$

From these two equations, at most two unknowns can be found uniquely. For serviceability limit states, a load factor of 1.0 is commonly used, and  $\alpha = 1$  will be used here. (Note: only the ratio  $\alpha/\phi$  need actually be determined for the settlement problem.)

Given  $\alpha/\phi$ ,  $\hat{P}$ ,  $\hat{E}$ , and  $H$ , Eq. (10) is relatively efficiently solved for  $B$  using 1-pt iteration;

$$[12] \quad B_{i+1} = 0.61 \left( 1 - e^{-1.18H/B_i} \right) \left( \frac{\alpha\hat{P}}{\delta_{max}\phi\hat{E}} \right)$$

for  $i = 1, 2, \dots$  until successive estimates of  $B$  are sufficiently similar. A reasonable starting guess is  $B_1 = 0.4(\alpha\hat{P})/(\delta_{max}\phi\hat{E})$ .

In Eq. (11), the random variables  $u_1$  and  $B$  are common to both sides of the inequality and so can be canceled. It will also be assumed that the footing load is lognormally distributed and that the estimated load,  $\hat{P}$ , equals the (non-random) *median load*, that is

$$[13] \quad \hat{P} = \exp\{\mu_{\ln P}\}$$

Setting the value of  $\hat{P}$  to the median load considerably simplifies the theory in the sequel, but it should be noted that the definition of  $\hat{P}$  will directly affect the magnitude of the estimated resistance factors. The lognormal distribution was selected because it results in loads which are strictly non-negative (uplift problems should be dealt with separately, and not handled via the tail end of a normal distribution assumption). The results to follow should be similar for any reasonable load distribution (e.g. Gamma, Chi-Square, etc) having the same mean and variance.

Collecting all remaining random quantities leads to the simplified design probability

$$[14] \quad \mathbf{P} \left[ P \frac{\hat{E}}{E_{eff}} > \frac{\alpha}{\phi} e^{\mu_{\ln P}} \right] = p_{max}$$

The estimated modulus,  $\hat{E}$ , and the effective elastic modulus,  $E_{eff}$ , will also be assumed to be lognormally distributed. Under these assumptions, if  $W$  is defined as

$$[15] \quad W = P \frac{\hat{E}}{E_{eff}}$$

then  $W$  is also lognormally distributed, so that

$$[16] \quad \ln W = \ln P + \ln \hat{E} - \ln E_{eff}$$

is normally distributed with mean

$$[17] \quad \mu_{\ln W} = \mu_{\ln P} + \mu_{\ln \hat{E}} - \mu_{\ln E_{eff}}$$

It is assumed that the load distribution is known, so that  $\mu_{\ln P}$  and  $\sigma_{\ln P}^2$  are known. The nature of the other two terms on the right hand side will now be investigated.

Assume that  $\hat{E}$  is estimated from a series of  $m$  soil samples that yield the observations  $E_1^o, E_2^o, \dots, E_m^o$ . To investigate the nature of this estimate, it is constructive to first consider the effective elastic modulus,  $E_{eff}$ , as seen by the footing. Analogous to the estimate for  $\hat{E}$ , it can be imagined that the soil volume under the footing is partitioned into a large number of soil ‘samples’ (although most of them, if not all, will remain unsampled),  $E_1, E_2, \dots, E_n$ . Investigations by Fenton and Griffiths (2002) suggest that the effective elastic modulus, as seen by the footing,  $E_{eff}$ , is a geometric average of the soil properties in the block under the footing, that is

$$[18] \quad E_{eff} = \left( \prod_{i=1}^n E_i \right)^{1/n} = \exp \left\{ \frac{1}{n} \sum_{i=1}^n \ln E_i \right\}$$

If  $\hat{E}$  is to be a good estimate of  $E_{eff}$ , which is desirable, then it should be similarly determined as a geometric average of the observed samples  $E_1^o, E_2^o, \dots, E_m^o$ ,

$$[19] \quad \hat{E} = \left( \prod_{j=1}^m E_j^o \right)^{1/m} = \exp \left\{ \frac{1}{m} \sum_{j=1}^m \ln E_j^o \right\}$$

since this estimate of  $E_{eff}$  is unbiased in the median, ie. the median of  $\hat{E}$  is equal to the median of  $E_{eff}$ . This is a fairly simple estimator, and no attempt is made here to account for the location of samples relative to the footing. Note that if the soil is layered horizontally and it is desired to specifically capture the layer information, then Eqs. 18 and 19 can be applied to each layer individually – the final  $\hat{E}$  and  $E_{eff}$  values are then computed as harmonic averages of the layer values. Although the distribution of a harmonic average is not simply defined, a lognormal approximation is often reasonable.

Under these definitions, the means of  $\mu_{\ln \hat{E}}$  and  $\mu_{\ln E_{eff}}$  are identical,

$$[20] \quad \mu_{\ln E_{eff}} = E [\ln E_{eff}] = \mu_{\ln E}$$

$$[21] \quad \mu_{\ln \hat{E}} = E [\ln \hat{E}] = \mu_{\ln E}$$

where  $\mu_{\ln E}$  is the mean of the logarithm of elastic moduli of any sample. Thus, as long as Equations (18) and (19) hold, the mean of  $\ln W$  simplifies to

$$[22] \quad \mu_{\ln W} = \mu_{\ln P}$$

Now, attention can be turned to the variance of  $\ln W$ . If the variability in the load  $P$  is independent of the soil's elastic modulus field then the variance of  $\ln W$  is

$$[23] \quad \sigma_{\ln W}^2 = \sigma_{\ln P}^2 + \sigma_{\ln \hat{E}}^2 + \sigma_{\ln E_{eff}}^2 - 2\text{Cov} [\ln \hat{E}, \ln E_{eff}]$$

The variances of  $\ln \hat{E}$  and  $\ln E_{eff}$  can be expressed in terms of the variance of  $\ln E$  using two variance reduction functions,  $\gamma^o$  and  $\gamma$ , defined as follows

$$[24a] \quad \gamma^o(m) = \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m \rho_{ij}^o$$

$$[24b] \quad \gamma(n) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij}$$

where  $\rho_{ij}^o$  is the correlation coefficient between  $\ln E_i^o$  and  $\ln E_j^o$  and  $\rho_{ij}$  is the correlation coefficient between  $\ln E_i$  and  $\ln E_j$ . These functions can be computed numerically once the locations of all soil ‘samples’ are known. Both  $\gamma^o(1)$  and  $\gamma(1)$  have value 1.0 when only 1 sample is used to specify  $\hat{E}$  or  $E_{eff}$ , respectively (when samples are ‘point’ samples then 1 sample corresponds to zero volume – however, in this paper, it is assumed that there is some representative sample volume from which the mean and variance of the elastic modulus field are estimated and this corresponds to the ‘point’ measure). As the number of samples increases the variance reduction function decreases towards zero at a rate inversely proportional to the total sample volume (see Vanmarcke 1984). If

the volume of the soil under the footing is  $B \times B \times H$  then a reasonable approximation to  $\gamma(n)$  is obtained by assuming a separable form;

$$[25] \quad \gamma(n) \simeq \gamma_1(2B/\theta_{\ln E})\gamma_1(2B/\theta_{\ln E})\gamma_1(2H/\theta_{\ln E})$$

where  $\gamma_1(a)$  is the 1-D variance function corresponding to a Markov correlation;

$$[26] \quad \gamma_1(a) = \frac{1}{a^2} [a + e^{-a} - 1]$$

As an aside, Fenton and Griffiths, 2002, suggest that the depth to the bedrock,  $H$ , be limited to no more than about  $10B$  in the calculation of the soil volume under the footing. However, the effective strain zone is generally quite a bit shallower, so a maximum depth of  $2B$  to  $4B$  might be more appropriate as a limitation on  $H$ .

An approximation to  $\gamma^o(m)$  is somewhat complicated by the fact that samples for  $\hat{E}$  are likely to be collected at separate locations. If the observations are sufficiently separated that they can be considered independent (eg. separated by more than  $\theta_{\ln E}$ ), then  $\gamma^o(m) = 1/m$ . If they are collected from within a contiguous volume,  $V^o$ , then

$$[27] \quad \gamma^o(m) \simeq \gamma_1(2R/\theta_{\ln E})\gamma_1(2R/\theta_{\ln E})\gamma_1(2H/\theta_{\ln E})$$

where the total plan area of soil sampled is  $R \times R$  (for example, a CPT sounding can probably be assumed to be sampling an effective area equal to about  $0.2 \times 0.2 \text{ m}^2$ , so that  $R = 0.2 \text{ m}$ ). The true variance reduction function will be somewhere in between. In this paper, the soil is sampled by examining one or more columns of the finite element model, and so for an individual column,  $R \times R$  becomes replaced by,  $\Delta x \times \Delta y$ , which are the plan dimensions of the finite elements and Eq. (27) can be used to obtain the variance reduction function for a single column. If more than one column is sampled, then

$$[28] \quad \gamma^o(m) \simeq \frac{\gamma_1(2\Delta x/\theta_{\ln E})\gamma_1(2\Delta y/\theta_{\ln E})\gamma_1(2H/\theta_{\ln E})}{n_{eff}}$$

where  $n_{eff}$  is the effective number of independent columns sampled. If the sampled columns are well separated (ie, by more than the correlation length), then they could be considered independent, and  $n_{eff}$  would be equal to the number of columns sampled. If the columns are closely clustered (relative to the correlation length), then  $n_{eff}$  would decrease towards 1. The actual number is somewhere in between and can be estimated by judgement.

With these results,

$$[29a] \quad \sigma_{\ln \hat{E}}^2 = \gamma^o(m)\sigma_{\ln E}^2$$

$$[29b] \quad \sigma_{\ln E_{eff}}^2 = \gamma(n)\sigma_{\ln E}^2$$

The covariance term in Eq. (23) is computed from

$$\text{Cov} [\ln \hat{E}, \ln E_{eff}] = \frac{1}{mn} \sum_{j=1}^m \sum_{i=1}^n \text{Cov} [\ln E_j^o, \ln E_i]$$

$$\begin{aligned}
[30] \quad &= \sigma_{\ln E}^2 \left[ \frac{1}{mn} \sum_{j=1}^m \sum_{i=1}^n \rho'_{ij} \right] \\
&= \sigma_{\ln E}^2 \rho'_{ave}
\end{aligned}$$

where  $\rho'_{ij}$  is the correlation coefficient between  $\ln E_j^o$  and  $\ln E_i$  and  $\rho'_{ave}$  is the average of all these correlations. If the estimate,  $\ln \hat{E}$ , is to be at all useful in a design, the value of  $\rho'_{ave}$  should be reasonably high. However, its magnitude depends on the degree of spatial correlation (measured by  $\theta_{\ln E}$ ) and the distance between the observations  $E_i^o$  and the soil volume under the footing. The correlation function of Eq. (2) captures both of these effects. That is, there will exist an ‘average’ distance  $\tau'_{ave}$  such that

$$[31] \quad \rho'_{ave} = \exp \left\{ \frac{-2\tau'_{ave}}{\theta_{\ln E}} \right\}$$

and the problem is to find a reasonable approximation to  $\tau'_{ave}$  if the numerical calculation of Eq. (30) is to be avoided. The approximation considered in this study is that  $\tau'_{ave}$  is defined as the average absolute distance between the  $E_i^o$  samples and a vertical line below the center of the footing, with a sample taken anywhere under the footing to be considered to be taken at the footing corner (eg, at a distance  $B/\sqrt{2}$  from the centerline) – this latter restriction is taken to avoid a perfect correlation when a sample is taken directly at the footing centerline, which would be incorrect. In addition, a side study performed by the authors, which is not reported here, indicated that for all moderate correlation lengths ( $\theta_{\ln E}$  of the order of the footing width) the true  $\tau'_{ave}$  differed by less than about 10% from the approximation  $B/\sqrt{2}$  for any sample taken under the footing.

Using these definitions, the variance of  $\ln W$  can be written

$$\begin{aligned}
[32] \quad \sigma_{\ln W}^2 &= \sigma_{\ln P}^2 + \sigma_{\ln E}^2 [\gamma^o(m) + \gamma(n) - 2\rho'_{ave}] \\
&\geq \sigma_{\ln P}^2
\end{aligned}$$

The limitation  $\sigma_{\ln W}^2 \geq \sigma_{\ln P}^2$  is introduced because it is possible, using the approximations suggested above, for the quantity inside the square brackets to become negative, which is physically inadmissible. It is assumed that if this happens that the sampling has reduced the uncertainty in the elastic modulus field essentially to zero.

With these results in mind the design probability becomes

$$\begin{aligned}
\mathbf{P} \left[ P \frac{\hat{E}}{E_{eff}} > \frac{\alpha}{\phi} e^{\mu_{\ln P}} \right] &= \mathbf{P} \left[ W > \frac{\alpha}{\phi} e^{\mu_{\ln P}} \right] \\
&= \mathbf{P} [\ln W > \ln \alpha - \ln \phi + \mu_{\ln P}] \\
[33] \quad &= 1 - \Phi \left( \frac{-\ln \phi}{\sigma_{\ln W}} \right) \quad (\text{assuming } \alpha = 1) \\
&= p_{max}
\end{aligned}$$

from which the required resistance factor,  $\phi$ , can be found as

$$[34] \quad \phi = \exp \{ -z_{p_{max}} \sigma_{\ln W} \}$$

where  $z_{p_{max}}$  is the point on the standard normal distribution having exceedance probability  $p_{max}$ . For  $p_{max} = 0.05$ , which will be assumed in this paper,  $z_{p_{max}} = 1.645$ .

It is instructive at this point to consider a limiting case, namely where  $\hat{E}$  is a perfect estimate of  $E_{eff}$ . In this case,  $\hat{E} = E_{eff}$ , which implies that  $m = n$  and the observations  $E_1^o, \dots$  coincide identically with the ‘samples’  $E_1, \dots$ . In this case,  $\gamma^o = \gamma$  and  $\rho = 1$ , so that

$$[35] \quad \sigma_{\ln W}^2 = \sigma_{\ln P}^2$$

from which the required resistance factor can be calculated as

$$[36] \quad \phi = \exp \left\{ -z_{p_{max}} \cdot \sigma_{\ln P} \right\}$$

For example, if  $p_{max} = 0.05$  and the coefficient of variation of the load is  $V_P = 0.1$ , then  $\phi = 0.85$ . Alternatively, for the same maximum acceptable failure probability, if  $V_P = 0.3$ , then  $\phi$  decreases to 0.62.

One difficulty with the computation of  $\sigma_{\ln E_{eff}}^2$ , that is apparent in the approximation of Eq. (25), is that it depends on the footing dimension  $B$ . From the point of view of the design probability, Eq. (14), this means that  $B$  does not entirely disappear, and the equation is still interpreted as the probability that a footing of a certain size will fail to stay within the serviceability limit state. The major implication of this interpretation is that if Eq. (14) is used conditionally to determine  $\phi$ , then the design resistance factor,  $\phi$ , will have some dependence on the footing size – this is not convenient for a design code (imagine designing a concrete beam if  $\phi_c$  varied with the beam dimension). Thus, as is, Eq. (14) should be used conditionally to determine the reliability of a footing against settlement failure once it has been designed. The determination of  $\phi$  must then proceed by using the total probability theorem; that is, find  $\phi$  such that

$$[37] \quad p_{max} = \int_0^\infty \mathbf{P} \left[ W > \frac{\alpha}{\phi} \hat{P} \mid B \right] f_B(b) db$$

where  $f_B$  is the probability distribution of the footing width  $B$ . The distribution of  $B$  is not easily obtained – it is a function of  $H$ ,  $\hat{P}$ ,  $\delta_{max}$ , the parameters of  $\hat{E}$ , and the load and resistance factors,  $\alpha$  and  $\phi$ , see Eq. (12) – and so the value of  $\phi$  is not easily determined using Eq. (37). One possible solution is to assume that changes in  $B$  do not have a great influence on the computed value of  $\phi$  and to take  $B = B_{med}$ , where  $B_{med}$  is the (non-random) footing width required by the median elastic modulus using a moderate resistance factor of  $\phi = 0.5$  in Eq. (12). This approach will be adopted in this paper, and will be validated by the simulation to be discussed next.

#### 4. Design Simulations

As mentioned above, the resistance factor  $\phi$  cannot be directly obtained by solving Eq. (14), for given  $B$ , simultaneously with Eq. (10) since this would result in a resistance factor which depends on the footing dimension. To find the value of  $\phi$  to be used for any footing size involves solving Eq. (37). Unfortunately, this is not feasible since the distribution of  $B$  is unknown (or, at least very difficult to compute). A simple solution is to use Monte Carlo simulation to estimate the probability on the right hand side of Eq. (37) and then use the simulation results to assess the validity of the simplifying assumption that  $B_{med}$  can be used to find  $\phi$  using Eq. (14). In this paper, the Random Finite Element Method (RFEM) will be employed within a design context to perform the desired simulation. The approach is described as follows;

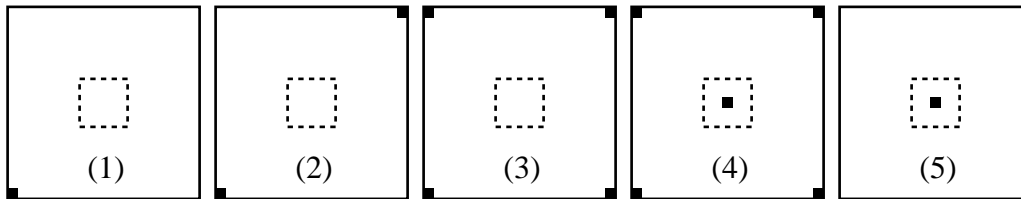
- 1) decide on a maximum tolerable settlement,  $\delta_{max}$ . In this paper,  $\delta_{max} = 0.025$  m.
- 2) estimate the nominal footing load,  $\hat{P}$ , to be the median load applied to the footing by the supported structure (it is assumed that the load distribution is known well enough to know its median,  $\hat{P} = e^{\mu_{\ln P}}$ ).
- 3) simulate an elastic modulus field,  $E(x)$ , for the soil from a lognormal distribution with specified mean,  $\mu_E$ , variance,  $\sigma_E^2$ , and correlation structure (Eq. 2) with correlation length  $\theta_{\ln E}$ . The field is simulated using the Local Average Subdivision (LAS) method (Fenton, 1990) whose local average values are assigned to corresponding finite elements.
- 4) ‘virtually’ sample the soil to obtain an estimate,  $\hat{E}$ , of its elastic modulus. In a real site investigation, the geotechnical engineer may estimate the soil’s elastic modulus and depth to firm stratum by performing one or more CPT or SPT soundings. In this simulation, one or more vertical columns of the soil model are selected to yield the elastic modulus samples. That is,  $\hat{E}$  is estimated using a geometric average, Eq. (19), where  $E_1^o$  is the elastic modulus of the top element of a column,  $E_2^o$  is the elastic modulus of the 2nd to top element of the same column, and so on to the base of the column. One or more columns may be included in the estimate, as will be discussed shortly, and measurement and model errors are not included in the estimate – the measurements are assumed precise.
- 5) letting  $\delta_p = \delta_{max}$ , and for given factors  $\alpha$  and  $\phi$  solve Eq. (12) for  $B$ . This constitutes the footing design. Note that design widths are normally rounded up to the next most easily measured dimension (eg 1684 mm would probably be rounded up to 1700 mm). In the same way, the design value of  $B$  is rounded up to the next larger element boundary, since the finite element model assumes footings are a whole number of elements wide. (The finite element model uses elements which are 0.15 m wide, so  $B$  is rounded up to the next larger multiple of 0.15 m.)
- 6) simulate a lognormally distributed footing load,  $P$ , having median  $\hat{P}$  and variance  $\sigma_P^2$ .
- 7) compute the ‘actual’ settlement,  $\delta$ , of a footing of width  $B$  under load  $P$  on a random elastic modulus field using the finite element model. In this step, the virtually sampled random field generated in step (3) above is mapped to the finite element mesh, the footing of width  $B$  (suitably rounded up to a whole number of elements wide) is placed on the surface and the settlement computed by finite element analysis.
- 8) if  $\delta > \delta_{max}$ , the footing design is assumed to have failed.
- 9) repeat from step (3) a large number of times ( $n = 1000$ , in this paper), counting the number of footings,  $n_f$ , which experienced a design failure. The failure probability is then estimated as  $\hat{p}_f = n_f/n$ .

By repeating the entire process over a range of possible values of  $\phi$  the resistance factor which leads to an acceptable probability of failure,  $p_f = p_{max}$ , can be selected. This ‘optimal’ resistance factor will also depend on;

- 1) the number and locations of sampled columns (analogous to the number and locations of CPT/SPT soundings),
- 2) the coefficient of variation of the soil’s elastic modulus,  $V_E$ ,
- 3) the correlation length,  $\theta_{ln E}$ ,

and the simulation will be repeated over a range of values of these parameters to see how they affect  $\phi$ .

Five different sampling schemes will be considered in this study, as illustrated in Figure 4. The outer solid line denotes the edge of the soil model, and the interior dashed line the location of the footing. The small black squares show the plan locations where the site is virtually sampled. It is expected that the quality of the estimate of  $E_{eff}$  will improve for higher numbered sampling schemes. That is, the probability of design failure will decrease for higher numbered sampling schemes, everything else being held constant.



**Figure 4.** Sampling schemes considered in this study.

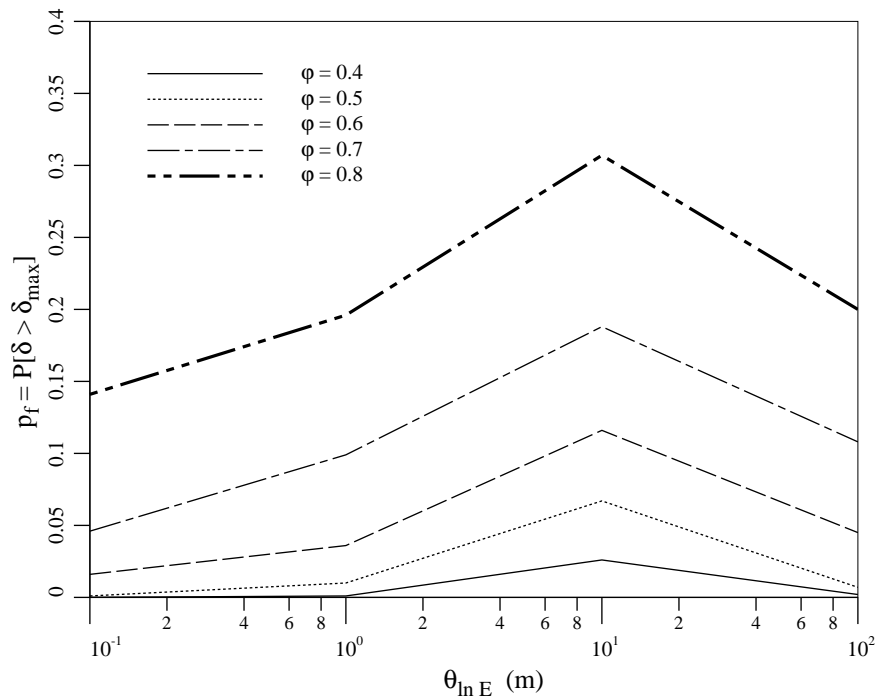
Table 1 lists the other parameters, aside from sampling schemes, varied in this study. In total 300 RFEM runs, each involving 1000 realizations were performed. Based on 1000 independent realizations, the estimated failure probability,  $\hat{p}_f$ , has standard error  $\sqrt{\hat{p}_f(1 - \hat{p}_f)/1000}$ , which for a probability level of 5% is 0.7%.

**Table 1.** Input parameters varied in the study while holding  $H = 4.8$  m,  $D = 9.6$  m,  $\mu_P = 1200$  kN,  $V_P = 0.25$ ,  $\mu_E = 20$  MPa, and  $\nu = 0.3$  constant.

Parameter	Values Considered
$V_E$	0.1, 0.2, 0.5
$\theta_{ln E}(m)$	0.1, 1.0, 10.0, 100.0
$\phi$	0.4, 0.5, 0.6, 0.7, 0.8

## 5. Simulation Results

Figure 5 shows the effect of the correlation length on the probability of failure for sampling scheme #1 (a single sampled column at the corner of site) and for  $V_E = 0.5$ . The other sampling schemes and values of  $V_E$  displayed similarly shaped curves. Of particular note in Figure 5 is the fact that the probability of failure reaches a maximum for an intermediate correlation length, in this case when  $\theta_{\ln E} \simeq 10$  m. This is as expected, since for stationary random fields the values of  $\hat{E}$  and  $E_{eff}$  will coincide for both vanishingly small correlation lengths (where local averaging results in both becoming equal to the median) and for very large correlation lengths (where  $\hat{E}$  and  $E_{eff}$  become perfectly correlated) and so the largest differences between  $\hat{E}$  and  $E_{eff}$  will occur at intermediate correlation lengths. The true maximum could lie somewhere between  $\theta_{\ln E} = 1$  m and  $\theta_{\ln E} = 100$  m in this particular study.

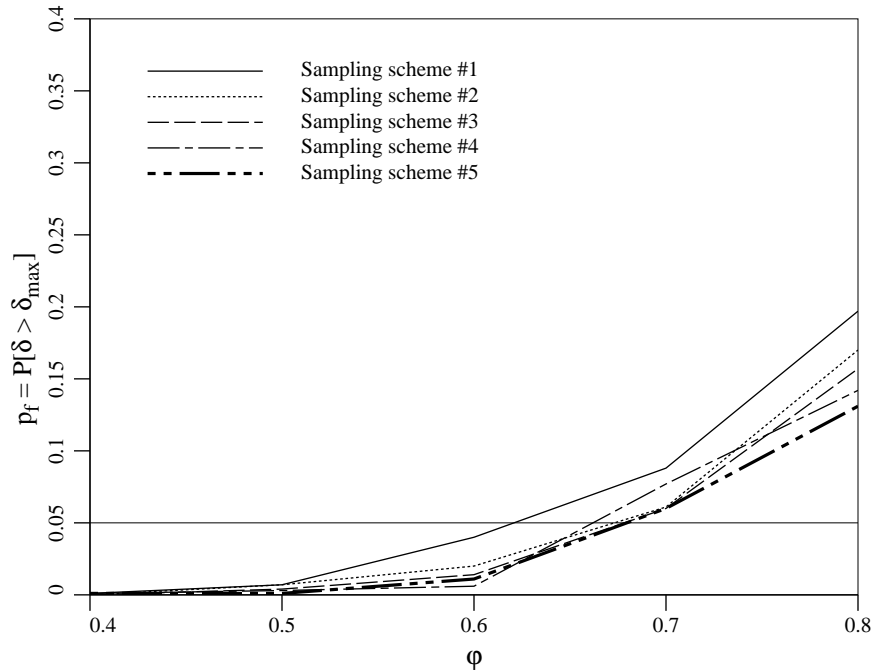


**Figure 5.** Effect of correlation length,  $\theta_{\ln E}$ , on probability of failure,  $p_f = P[\delta > \delta_{max}]$ .

Where the maximum lies for arbitrary sampling patterns is still unknown, but the authors expect that it is probably safe to say that taking  $\theta_{\ln E}$  approximately equal to the average distance between sample locations and the footing center (but not less than approximately the footing size) would yield suitably conservative failure probabilities. In this paper, the  $\theta_{\ln E} = 10$  m results will be concentrated on since these yielded the most conservative designs in this study.

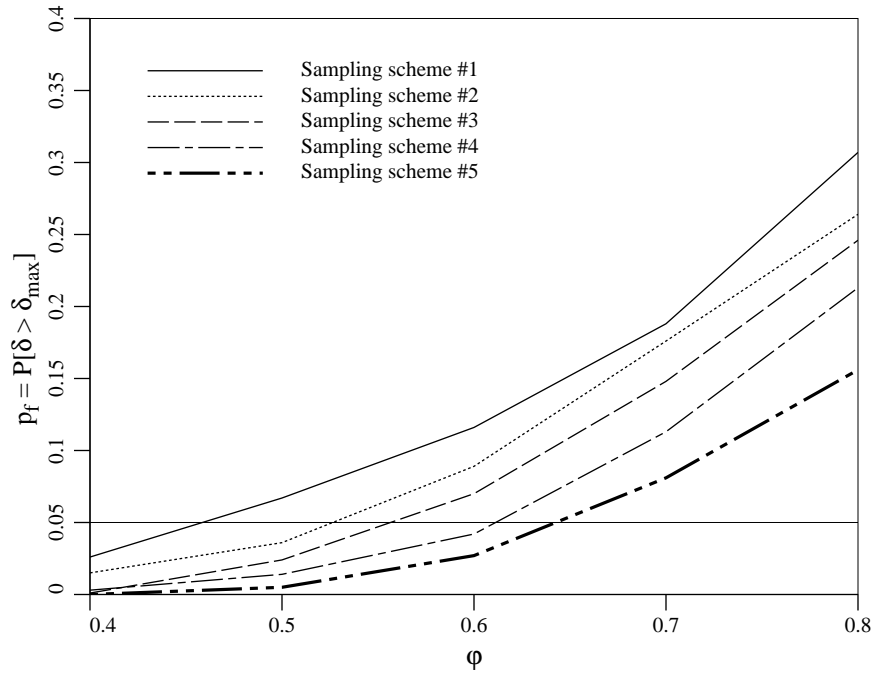
Figure 6 shows how the estimated probability of failure varies with resistance factor for the five sampling schemes considered with  $V_E = 0.2$  and  $\theta_{\ln E} = 10$  m. This Figure can be used for design by drawing a horizontal line across at the target probability,  $p_{max}$  – to illustrate this, a light line has been drawn across at  $p_{max} = 0.05$  – and then reading off the required resistance factor for a given sampling scheme. For  $p_{max} = 0.05$ , it can be seen that  $\phi \simeq 0.62$  for the ‘worst case’ sampling scheme #1. For all the other sampling schemes considered, the required resistance factor

is between about 0.67 and 0.69. Because the standard error of the estimated  $p_f$  values is 0.7% at this level, the relative positions of the lines tends to be somewhat erratic. What Figure 6 is saying, essentially, is that at low levels of variability, increasing the number of samples does not greatly affect the probability of failure.



**Figure 6.** Effect of resistance factor,  $\phi$ , on probability of failure,  $p_f = P[\delta > \delta_{max}]$  for  $V_E = 0.2$  and  $\theta_{ln E} = 10$  m.

When the coefficient of variation,  $V_E$ , increases the distinction between sampling schemes becomes more pronounced. Figure 7 shows the failure probability for the various sampling schemes at  $V_E = 0.5$  and  $\theta_{ln E} = 10$  m. Improved sampling now makes a significant difference to the required value of  $\phi$ , which ranges from  $\phi \simeq 0.46$  for sampling scheme #1 to  $\phi \simeq 0.65$  for sampling scheme #5, assuming a target probability of  $p_{max} = 0.05$ . The implications of Figure 7 are that when soil variability is significant, considerable design/construction savings can be achieved when the sampling scheme is improved.



**Figure 7.** Effect of resistance factor,  $\phi$ , on probability of failure,  $p_f = P[\delta > \delta_{max}]$  for  $V_E = 0.5$  and  $\theta_{ln E} = 10$  m.

The approximation to the analytical expression for the failure probability can now be evaluated. For the case considered in Figure 7,  $V_E = 0.5$  and  $V_P = 0.25$  so that

$$\begin{aligned}\sigma_{\ln E}^2 &= \ln(1 + V_E^2) = 0.2231 \\ \sigma_{\ln P}^2 &= \ln(1 + V_P^2) = 0.0606\end{aligned}$$

To compute the variance reduction function,  $\gamma(n)$ , the footing width corresponding to the median elastic modulus is needed. For this calculation, an initial value of  $\phi$  is also needed, and the moderate value of  $\phi = 0.5$  is recommended. For  $\mu_E = 20000$  kPa, the median elastic modulus,  $\tilde{E}$ , is

$$\tilde{E} = \frac{\mu_E}{\sqrt{1 + V_E^2}} = \frac{20000}{\sqrt{1 + 0.5^2}} = 17889 \text{ kPa}$$

and for  $\mu_P = 1200$  kN, the median footing load is

$$\hat{P} = \frac{\mu_P}{\sqrt{1 + V_P^2}} = \frac{1200}{\sqrt{1 + 0.25^2}} = 1164.2 \text{ kN}$$

Solving Eq. (12) iteratively gives  $B_{med} = 2.766$  m. The corresponding variance reduction factors are

$$\begin{aligned}\gamma_1\left(\frac{2(4.8)}{10}\right) &= \frac{1}{0.96^2} [0.96 + e^{-0.96} - 1] = 0.74413 \\ \gamma_1\left(\frac{2(2.766)}{10}\right) &= \frac{1}{0.5532^2} [0.5532 + e^{-0.5532} - 1] = 0.83852\end{aligned}$$

which gives

$$\gamma(n) \simeq (0.83852)^2(0.74413) = 0.5232$$

Now consider sampling scheme #1 which involve a single vertical sample with  $R = \Delta x = 0.15$  m and corresponding variance reduction factor,

$$\gamma_1 \left( \frac{2(0.15)}{10} \right) = \frac{1}{0.03^2} [0.03 + e^{-0.03} - 1] = 0.99007$$

$$\gamma^o(m) \simeq (0.99007)^2(0.74413) = 0.7294$$

For sampling scheme #1,  $\tau'_{ave} \simeq \sqrt{2}(9.6/2) = 6.79$  m is the (approximate) distance from the sample point to the center of the footing. In this case,

$$\rho'_{ave} = \exp \left\{ -\frac{2(6.79)}{10} \right\} = 0.2572$$

which gives us, using Eq. (32),

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.7294 + 0.5232 - 2(0.2572)] = 0.2253$$

so that  $\sigma_{\ln w} = 0.4746$ . For  $z_{0.05} = 1.645$ , the required resistance factor is determined by Eq. (34) to be

$$\phi = \exp\{-1.645(0.4746)\} = 0.46$$

The corresponding value on Figure 7 is also 0.46. Although this agreement is excellent, it must be remembered that this is an approximation, and the precise agreement may be due somewhat to mutually cancelling errors and to chance, since the simulation estimates are themselves somewhat random. For example, if the more precise formulas of Eq's (24a), (24b), and (30) are used then  $\gamma^o(m) = 0.7432$ ,  $\gamma(n) = 0.6392$ , and  $\rho'_{ave} = 0.2498$ , which gives

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.7432 + 0.6392 - 2(0.2498)] = 0.2576$$

so that the 'more precise' required resistance factor actually has poorer agreement with simulation;

$$\phi = \exp\{-1.645\sqrt{0.2576}\} = 0.43$$

It is also to be remembered that the 'more precise' result above is still conditioned on  $B = B_{med}$  and  $\phi = 0.5$ , whereas the simulation results are unconditional. Nevertheless, these results suggest that the approximations are insensitive to variations in  $B$  and  $\phi$ , and are thus reasonably general.

Sampling scheme #2 involves two sampled columns separated by more than  $\theta_{\ln E} = 10$  m so that  $n_{eff}$  can be taken as 2. This means that  $\gamma^o(m) \simeq 0.7294/2 = 0.3647$ . The average distance from the footing centerline to the sampled columns is still about 6.79 m, so that  $\rho'_{ave} = 0.2572$ . Now

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.3647 + 0.5232 - 2(0.2572)] = 0.1439$$

and the required resistance factor is

$$\phi = \exp\{-1.645\sqrt{0.1439}\} = 0.54$$

The corresponding value on Figure 7 is about 0.53.

Sampling scheme #3 involves four sampled columns, separated by somewhat less than  $\theta_{\ln E} = 10$  m. Due to the resulting correlation between columns,  $n_{eff} \simeq 3$  is selected (ie. somewhat less than the ‘independent’ value of 4). This gives  $\gamma^o(m) \simeq 0.7294/3 = 0.2431$ . Since the average distance from the footing centerline to the sample columns is still about 6.79 m,

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.2431 + 0.5232 - 2(0.2572)] = 0.1268$$

The required resistance factor is

$$\phi = \exp\{-1.645\sqrt{0.1268}\} = 0.57$$

The corresponding value on Figure 7 is about 0.56.

Sampling scheme #4 involves 5 sampled columns, also separated by somewhat less than  $\theta_{\ln E} = 10$  m and  $n_{eff} \simeq 4$  is selected to give  $\gamma^o(m) \simeq 0.7294/4 = 0.1824$ . One of the sampled columns lies below the footing, and so its ‘distance’ to the footing centerline is taken to be  $B_{med}/\sqrt{2} = 2.766/\sqrt{2} = 1.96$  m to avoid complete correlation. The average distance to sampling points is thus

$$\tau'_{ave} = \frac{4}{5}(6.79) + \frac{1}{5}(1.96) = 5.82$$

so that  $\rho'_{ave} = 0.3120$ . This gives

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.1824 + 0.5232 - 2(0.3120)] = 0.0788$$

The required resistance factor is

$$\phi = \exp\{-1.645\sqrt{0.0788}\} = 0.63$$

The corresponding value on Figure 7 is about 0.62.

For sampling scheme #5, the distance from the sample point to the center of the footing is zero, so  $\tau'_{ave}$  is taken to equal the distance to the footing corner,  $\tau'_{ave} = (2.766)/\sqrt{2} = 1.96$  m, as recommended earlier. This gives  $\rho'_{ave} = 0.676$  and

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.7294 + 0.5232 - 2(0.676)] = 0.0606 + 0.2231 [-0.0994] \rightarrow 0.0606$$

where approximation errors led to a negative variance contribution from the elastic modulus field which was ignored (ie. set to zero). In this case, the sampled information is deemed sufficient to render uncertainties in the elastic modulus negligible, so that  $\hat{E} \simeq E_{eff}$  and

$$\phi = \exp\{-1.645\sqrt{0.0606}\} = 0.67$$

The value of  $\phi$  read from Figure 7 is about 0.65. If the more precise formulas for the variance reduction functions and covariance terms are used, then  $\gamma^o(m) = 0.7432$ ,  $\gamma(n) = 0.6392$ , and  $\rho'_{ave} = 0.6748$ , which gives

$$\sigma_{\ln w}^2 = 0.0606 + 0.2231 [0.7432 + 0.6392 - 2(0.6748)] = 0.0679$$

Notice that this is very similar to the approximate result obtained above, which suggests that the assumption that samples taken below the footing largely eliminate uncertainty in the effective elastic modulus is reasonable. For this more accurate result,

$$\phi = \exp\{-1.645\sqrt{0.0679}\} = 0.65$$

which is the same as the simulation results.

Perhaps surprisingly, sampling scheme #5 outperforms, in terms of failure probability and resistance factor, sampling scheme #4, even though sampling scheme #4 involves considerably more information. The reason for this is that the good information taken below the footing is diluted by poorer information taken from farther away. This implies that when a sample is taken below the footing, other samples taken from farther away should be downweighted.

The computations illustrated above for all five sampling schemes can be summarized as follows;

- 1) Decide on an acceptable maximum settlement,  $\delta_{max}$ . Since serviceability problems in a structure usually arise as a result of differential settlement, rather than settlement itself, the choice of an acceptable maximum settlement is usually made assuming that differential settlement will be less than the total settlement of any single footing (see, eg. D'Appolonia, 1968).
- 2) Choose statistical parameters of the elastic modulus field,  $\mu_E$ ,  $\sigma_E$ , and  $\theta_{\ln E}$ . The last can be the 'worst case' correlation length, suggested here to approximately equal the average distance between sample locations and the footing center, but not to be taken less than the median footing dimension. The values of  $\mu_E$  and  $\sigma_E$  can be estimated from site samples (although the effect of using estimated values of  $\mu_E$  and  $\sigma_E$  in these computations has not been investigated) or from the literature.
- 3) Use Eqs.(1) to compute the statistical parameters of  $\ln E$  and then compute the median  $\tilde{E} = \exp\{\mu_{\ln E}\}$ .
- 4) Choose statistical parameters for the load,  $\mu_P$  and  $\sigma_P$ , and use these to compute the mean and variance of  $\ln P$ . Set  $\hat{P} = \exp\{\mu_{\ln P}\}$ .
- 5) Using a moderate resistance factor,  $\phi = 0.5$ , and the median elastic modulus,  $\tilde{E}$ , compute the 'median' value of  $B$  using the 1-pt iteration of Eq. (12). Call this  $B_{med}$ .
- 6) Compute  $\gamma(n)$  using Eq. (25) (or Eq. 24b) with  $B = B_{med}$ .
- 7) Compute  $\gamma^o(m)$  using Eq. (28) (or Eq. 24a).
- 8) Compute  $\rho'_{ave}$  using Eq. (31) (or Eq. 30) after selecting a suitable value for  $\tau'_{ave}$  as the average absolute distance between the sample columns and the footing center (where distances are taken to be no less than the distance to the footing corner,  $B_{med}/\sqrt{2}$ ).
- 9) Compute  $\sigma_{\ln w}$  using Eq. (32).
- 10) Compute the required resistance factor,  $\phi$ , using Eq. (34).

## 6. Conclusions

The paper presents approximate relationships based on random field theory which can be used to estimate resistance factors for appropriate for the LRFD settlement design of shallow foundations. Some specific comments arising from this research are as follows;

- 1) Two assumptions deemed to have the most influence on the resistance factors estimated in this study are 1) that the nominal load used for design,  $\hat{P}$ , is the median load and 2) that the load factor,  $\alpha$ , is equal to 1.0. Changes in  $\alpha$  result in a linear change in the resistance factor, e.g.  $\phi' = \alpha\phi$ , where  $\phi$  is the resistance factor found in this study and  $\phi'$  is the resistance factor corresponding to an  $\alpha$  which is not equal to 1.0. Changes in  $\hat{P}$  (for example, if  $\hat{P}$  were taken as some other load exceedance percentile) would result in first order linear changes to  $\phi$ , but further study would be required to specify the actual effect on the resistance factor.
- 2) The resistance factors obtained in this study should be considered to be *upper bounds* since the additional uncertainties arising from measurement and model errors have not been considered. To some extent, these additional error sources can be accommodated here simply by using a value of  $V_E$  greater than would actually be true at a site. For example, if  $V_E = 0.35$  at a site, the effects of measurement and model error might be accommodated by using  $V_E = 0.5$  in the relationships presented here. This issue needs additional study, but Meyerhof's (1982) comment that a "performance factor of 0.7 should be used for adequate reliability of serviceability estimates" suggests that the results presented here are reasonable (possibly a little conservative at the  $V_E = 0.5$  level) for all sources of error.
- 3) The use of a 'median' footing width,  $B_{med}$ , derived using a median elastic modulus and moderate  $\phi = 0.5$  value, rather than by using the full  $B$  distribution in the computation of  $\gamma(n)$  appears to be quite reasonable. This is validated by the agreement between the simulation results (where  $B$  varies with each realization) and the results obtained using the approximate relationships (see previous Section).
- 4) The computation of a required resistance factor assumes that the uncertainty (eg,  $V_E$ ) is known. In fact, at a given site, all three parameters  $\mu_E$ ,  $V_E$ , and  $\theta_{ln E}$  will be unknown and only estimated to various levels of precision by sampled data. To establish a LRFD code, at least  $V_E$  and  $\theta_{ln E}$  need to be known *a priori*. One of the significant results of this research is that a worst-case correlation length exists, which can be used in the development of a design code. While, the value of  $\sigma_E^2$  remains an outstanding issue, calibration with existing codes may very well allow its 'practical' estimation.
- 5) At low uncertainty levels, that is when  $V_E \leq 0.2$  or so, there is not much advantage to be gained by taking more than two sampled columns (eg. SPT or CPT borings) *in the vicinity of the footing*, as seen in Figure 6. This statement assumes that the soil is *stationary*. The assumption of stationarity implies that samples taken in one location are as good an estimator of the mean, variance, etc., as samples taken elsewhere. Since this is rarely true of soils, the qualifier "in the vicinity" was added to the above statement.
- 6) Although sampling scheme #4 involved five sampled columns and sampling scheme #5 involved only one sampled column, sampling scheme #5 outperformed #4. This is because the distance to the samples was not considered in the calculation of  $\hat{E}$ . Thus, in sampling scheme #4 the good estimate taken under the footing was diluted by four poorer estimates taken some distance away. Whenever a soil is sampled directly under a footing, the sample results should

be given much higher precedence than soil samples taken elsewhere. That is, the concepts of Best Linear Unbiased Estimation (BLUE), which takes into account the correlation between estimate and observation, should be used. In this paper a straightforward geometric average was used (arithmetic average of logarithms in log-space) for simplicity. Further work on the effect of the form of the estimator on the required resistance factor is needed.

## 7. List of Symbols

The following symbols are used in this paper:

- $B$  = footing width, as designed  
 $B_{med}$  = footing width required on median elastic modulus using moderate resistance factor  
 $D$  = plan width of soil model (= 9.6 m in this study)  
 $E$  = elastic modulus  
 $E(x_i)$  = elastic modulus at the spatial location  $x_i$   
 $\hat{E}$  = estimate of effective elastic modulus, derived from soil samples  
 $E_{eff}$  = effective uniform elastic modulus that, if underlying the footing, would yield the same settlement as actual  
 $E_i$  = one of  $n$  elastic modulus ‘samples’ forming a partition in the region under the footing  
 $E_j^o$  = one of  $m$  elastic modulus soil samples actually observed  
 $f_B$  = footing width probability density function  
 $G(x)$  = standard normal (Gaussian) random field  
 $H$  = overall depth of soil layer  
 $n$  = number of simulations  
 $n_f$  = number of simulations resulting in failure ( $\delta > \delta_{max}$ )  
 $n_{eff}$  = effective number of *independent* sampled soil columns  
 $P$  = actual applied footing load  
 $\hat{P}$  = median applied footing load  
 $p_f$  = probability of failure ( $\delta > \delta_{max}$ )  
 $\hat{p}_f$  = estimated probability of failure  
 $p_{max}$  = maximum acceptable probability of failure  
 $\hat{q}$  = estimated soil stress applied by footing  
 $R$  = sample plan dimension  
 $u_1$  = settlement influence factor  
 $V_E$  = elastic modulus coefficient of variation ( $\mu_E/\sigma_E$ )  
 $V_P$  = load coefficient of variation ( $\mu_P/\sigma_P$ )  
 $W = P\hat{E}/E_{eff}$   
 $x$  = spatial coordinate or position  
 $y$  = horizontal component of spatial position  
 $z$  = vertical component of spatial position  
 $z_{p_{max}}$  = point on standard normal distribution with exceedance probability  $p_{max}$   
 $\alpha$  = load factor  
 $\gamma$  = variance reduction function (due to local averaging)  
 $\gamma^o$  = variance reduction function for observed samples

$\gamma_1$  = 1-dimensional variance reduction function for Markov correlation  
 $\delta$  = footing settlement, positive downwards  
 $\delta_p$  = predicted footing settlement  
 $\delta_{max}$  = maximum acceptable footing settlement  
 $\theta_{\ln E}$  = isotropic correlation length of the log-elastic modulus field  
 $\mu_E$  = mean elastic modulus  
 $\mu_{\ln E}$  = mean of log-elastic modulus  
 $\mu_{\ln \hat{E}}$  = mean of the logarithm of the estimated effective elastic modulus  
 $\mu_{\ln E_{eff}}$  = mean of the logarithm of the effective elastic modulus underlying the footing  
 $\mu_P$  = mean footing load  
 $\mu_{\ln P}$  = mean of the log-footing load  
 $\mu_{\ln W}$  = mean of  $\ln W$   
 $\Phi$  = standard normal cumulative distribution function  
 $\phi$  = resistance factor  
 $\nu$  = Poisson's ratio  
 $\rho$  = correlation coefficient  
 $\rho_{ij}$  = correlation coefficient between  $\ln E_i$  and  $\ln E_j$   
 $\rho_{ij}^o$  = correlation coefficient between  $\ln E_i^o$  and  $\ln E_j^o$   
 $\rho'_{ij}$  = correlation coefficient between  $\ln E_i$  and  $\ln E_j^o$   
 $\rho'_{ave}$  = average correlation coefficient between  $\ln E_i$  and  $\ln E_j^o$   
 $\sigma_E$  = standard deviation of elastic modulus  
 $\sigma_{\ln E}$  = standard deviation of log-elastic modulus  
 $\sigma_{\ln \hat{E}}$  = standard deviation of the logarithm of the estimated effective elastic modulus  
 $\sigma_{\ln E_{eff}}$  = standard deviation of the logarithm of the effective elastic modulus underlying the footing  
 $\sigma_{\ln P}$  = standard deviation of the log-footing load  
 $\mathcal{T}$  = spatial lag vector  
 $\tau$  = lag distance, equal to  $|\mathcal{T}|$   
 $\tau'_{ave}$  = average distance between the footing center and the sampled soil columns

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